Designation: D4015 – 21

Standard Test Methods for Modulus and Damping of Soils by Fixed-Base Resonant Column Devices[1]

1. Scope*

1.1 These test methods cover the determination of shear modulus and shear damping as a function of shear strain amplitude for solid cylindrical specimens of soil in intact and reconstituted conditions by torsional vibration using resonant column devices. The vibration of the specimen may be superposed on a controlled static state of stress in the specimen. The vibration apparatus and specimen may be enclosed in a triaxial chamber and subjected to an all-around pressure and axial load. In addition, the specimen may be subjected to other controlled conditions (for example, pore-water pressure, degree of saturation, temperature). These test methods of modulus and damping determination are considered nondestructive when the shear strain amplitudes of vibration are less than 10^{-2} % (10^{-4} in./in.), and many measurements may be made on the same specimen and with various states of static stress.

1.2 Two device configurations are covered by these test methods: Device Type 1 where a known torque is applied to the top of the specimen and the resulting rotational motion is measured at the top of the specimen, and Device Type 2 where an uncalibrated torque is applied to the top of the specimen and the torque transmitted through the specimen is measured by a torque transducer at the base of the specimen. For both types of devices, the torque is applied to the active end (usually top) of the specimen and the rotational motion also is measured at the active end of the specimen.

1.3 These test methods are limited to the determination of the shear modulus and shear damping, the necessary vibration, and specimen preparation procedures related to the vibration, etc., and do not cover the application, measurement, or control of the axial and lateral static normal stresses. The latter procedures may be covered by, but are not limited to, Test Method D2850, D4767, or D7181.

1.4 *Significant Digits*—All recorded and calculated values shall conform to the guide for significant digits and rounding established in Practice D6026.

1.4.1 The procedures used to specify how data are collected/recorded and calculated in this standard are regarded as the industry standard. In addition, they are representative of the significant digits that should generally be retained. The procedures used do not consider material variation, purpose for obtaining the data, special purpose studies, or any considerations for the user's objectives; and it is common practice to increase or reduce significant digits of reported data to be commensurate with these considerations. It is beyond the scope of this standard to consider significant digits used in analysis methods for engineering design.

1.4.2 Measurements made to more significant digits or better sensitivity than specified in this standard shall not be regarded a nonconformance with this standard.

1.5 *Units*—The values stated in SI units are to be regarded as standard. The values given in parentheses are mathematical conversions to inch-pound units, which are provided for information only and are not considered standard. Reporting of test results in units other than SI shall not be regarded as nonconformance with these test methods.

1.5.1 The converted inch-pound units use the gravitational system of units. In this system, the pound (lbf) represents a unit of force (weight), while the unit for mass is slugs. The converted slug unit is not given, unless dynamic (F = ma) calculations are involved.

1.5.2 It is common practice in the engineering/construction profession to concurrently use pounds to represent both a unit of mass (lbm) and of force (lbf). This implicitly combines two separate systems of units; that is, the absolute system and the gravitational system. It is scientifically undesirable to combine the use of two separate sets of inch-pound units within a single standard. As stated, this standard includes the gravitational system of inch-pound units and does not use/present the slug

[1] These test methods are under the jurisdiction of ASTM Committee D18 on Soil and Rock and are the direct responsibility of Subcommittee D18.09 on Cyclic and Dynamic Properties of Soils.

Current edition approved Dec. 1, 2021. Published December 2021. Originally approved in 1981. Last previous edition approved in 2015 as D4015 –15^{ε1}. DOI: 10.1520/D4015-21.

*A Summary of Changes section appears at the end of this standard unit for mass. However, the use of balances or scales recording pounds of mass (lbm) or recording density in lbm/ft^3 shall not be regarded as nonconformance with this standard.

1.6 *This standard does not purport to address all of the safety concerns, if any, associated with its use. It is the responsibility of the user of this standard to establish appropriate safety, health, and environmental practices and determine the applicability of regulatory limitations prior to use.*

1.7 *This international standard was developed in accordance with internationally recognized principles on standardization established in the Decision on Principles for the Development of International Standards, Guides and Recommendations issued by the World Trade Organization Technical Barriers to Trade (TBT) Committee.*

2. Referenced Documents

2.1 *ASTM Standards:*[2]

D653 Terminology Relating to Soil, Rock, and Contained Fluids

D2166/D2166M Test Method for Unconfined Compressive Strength of Cohesive Soil

D2216 Test Methods for Laboratory Determination of Water (Moisture) Content of Soil and Rock by Mass

D2850 Test Method for Unconsolidated-Undrained Triaxial Compression Test on Cohesive Soils

D3740 Practice for Minimum Requirements for Agencies Engaged in Testing and/or Inspection of Soil and Rock as Used in Engineering Design and Construction

D4753 Guide for Evaluating, Selecting, and Specifying Balances and Standard Masses for Use in Soil, Rock, and Construction Materials Testing

D4767 Test Method for Consolidated Undrained Triaxial Compression Test for Cohesive Soils

D6026 Practice for Using Significant Digits and Data Records in Geotechnical Data

D7181 Test Method for Consolidated Drained Triaxial Compression Test for Soils

3. Terminology

3.1 *Definitions*—For definitions of common technical terms used in this standard, refer to Terminology D653.

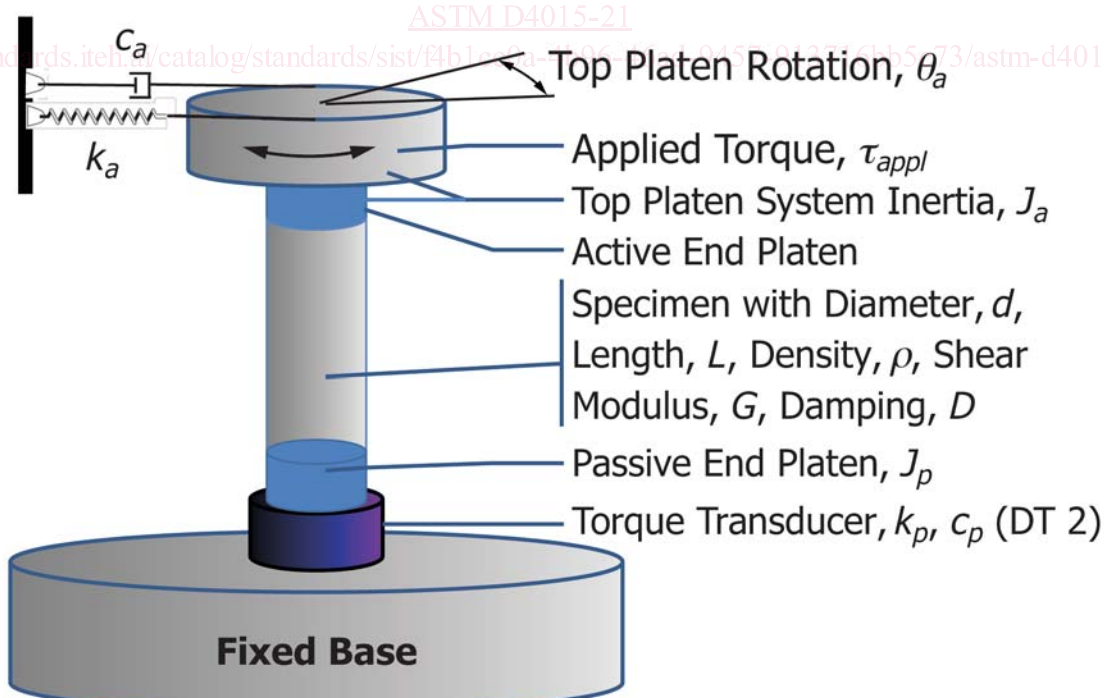
3.2 *Definitions of Terms Specific to This Standard:*

3.2.1 *damping capacity D* [unitless, typically expressed in %], *n—in resonant column systems*, is related to the component of the dynamic shear modulus that lags the applied shear stress by 90° degrees.

3.2.2 *Device Type 1, DT1, n—in resonant column systems*, a resonant column system as shown in Fig. 1 where the passive end platen is directly connected to the Fixed Base (no torque transducer), a calibrated vibratory torque is applied to the active end, and rotation is measured at the active end.

3.2.2.1 *Discussion*—The vibration excitation device may incorporate springs and dashpots connected to the active-end platen, where the spring constants and viscous damping coefficients must be known. The rotational inertia of the active-end

[2] For referenced ASTM standards, visit the ASTM website, www.astm.org, or contact ASTM Customer Service at service@astm.org. For *Annual Book of ASTM Standards* volume information, refer to the standard's Document Summary page on the ASTM website.



For Device Type 1, no torque transducer is needed and the Passive End Platen is connected to the Fixed Base.

FIG. 1 Resonant-Column Schematic for Both Device Types 1 and 2

platen and portions of the vibration excitation device moving with it must be known.

3.2.3 *Device Type 2, DT2, n—in resonant column systems*, a resonant column system as shown in Fig. 1 where the passive end platen is connected to a torque transducer, an uncalibrated torque is applied to the active end, torque is measured by the torque transducer at the passive end, and rotation is measured at the active end.

3.2.3.1 *Discussion*—The vibration excitation device may incorporate springs and dashpots connected to the active-end platen, but the spring constants and viscous damping coefficients are not needed. The rotational inertia of the active-end platen and portions of the vibration excitation device moving with it also are not needed.

3.2.4 *dimensionless specimen stiffness, DSS* [unitless], n—in resonant column systems*, is a complex number used to characterize the real and imaginary components of the specimen stiffness.

3.2.5 *dynamic shear modulus, G* [FL^{-2}], n—in resonant column systems*, is the ratio of shear stress to shear strain under vibratory conditions (also known as complex shear modulus).

3.2.6 *equivalent elastic shear modulus G [FL^{-2}], n—in resonant column systems*, is the component of the dynamic shear modulus that is in-phase with the applied shear stress.

3.2.7 *resonant-column system, n*—a system as shown in Fig. 1 consisting of a cylindrical specimen or column of soil enclosed with a flexible membrane that has platens attached to each end and where a sinusoidal vibration excitation device is attached to the active-end platen and where the other end is the passive-end platen that is rigidly fixed.

3.2.8 *specimen shear strain γ,* [unitless, frequently expressed as %], *n*—in resonant column systems, is the average shear strain in the specimen where the shear strain in each cross section varies from zero along the axis of rotation to a maximum at the perimeter of the specimen.

3.2.8.1 *Discussion*—The radius for calculating average shear strains vary depending on soil type, strain level, confining stress, etc. The default value of the radius for calculating average strain is 0.4*diameter but values in the range of 0.33 to 0.40*diameter may be used if the value is documented in the report.

3.2.9 *system resonant frequency f_r*[s^{-1}], *n—in resonant column systems*, for Device Type 1 is the lowest frequency at which the rotational velocity at the active end is in phase with the sinusoidal excitation torque and for Device Type 2, is the lowest frequency at which the rotational motion at the active end is a maximum.

4. Summary of Test Method

4.1 The resonant column device is shown schematically in Fig. 1. In the resonant column test, a cylindrical soil specimen, usually enclosed with a thin membrane, is subjected to an imposed static axial and lateral stress condition. Torsional sinusoidal vibrations are applied at the top of the soil specimen and the rotational response is measured. The frequency of excitation is varied until the system resonant frequency is achieved as described in 3.2.9. The devices may be operated at frequencies other than resonant frequencies. Given the geometry, mass and system parameters, the equivalent elastic shear modulus and damping capacity can be determined at a measured level of excitation vibration. The amplitude of vibration (which is related to shear strain) is typically varied to measure the variation of modulus and damping as a function of shear strain. The test is usually conducted at levels of shear strain between 0.00001 % and 0.2 %. (The upper limit of shear strain is dependent on the specimen stiffness and the maximum torque capability of the excitation system.) For specimens where the maximum shear strain measured is of the order of 0.01 %, the test is often conducted at several different sets of static axial and lateral stress conditions to measure the variation of moduli and damping with static stress states. The test results are dependent on sample quality/specimen disturbance which are beyond the scope of this standard.

5. Significance and Use

5.1 The equivalent elastic shear modulus and damping capacity of a given soil, as measured by the resonant column technique herein described, depend upon the strain amplitude of vibration, the state of effective stress, and the void ratio of the soil, temperature, time, etc. Since the application and control of the static axial and lateral stresses and the void ratio are not prescribed in these methods, the applicability of the results to field conditions will depend on the degree to which the application and control of the static axial and lateral stresses and the void ratio, as well as other parameters such as soil structure, duplicate field conditions. The techniques used to simulate field conditions depend on many factors and it is up to the engineer to decide on which techniques apply to a given situation and soil type. The results of these tests are useful for calculations involving soil-structure interaction and seismic response of soil deposits.

NOTE 1—The quality of the results produced by this standard is dependent on the competence of the personnel performing it, and the suitability of the equipment and facilities. Agencies that meet the criteria of Practice D3740 are generally considered capable of competent and objective testing/sampling/inspection/etc. Users of this standard are cautioned that compliance with Practice D3740 does not in itself assure reliable results. Reliable results depend on many factors; Practice D3740 provides a means of evaluating some of those factors.

6. Apparatus

6.1 *General*—The complete test apparatus is shown schematically in Fig. 1 and includes the platens for holding the specimen in the pressure cell, the vibration excitation device (torque motor), transducers for measuring the response, the control and readout instrumentation, and auxiliary equipment for specimen preparation. The theory for the resonant column is provided in Annex A1. The entire apparatus is generally enclosed within a pressure chamber (commonly referred to as a triaxial cell). For some apparatus that can apply an axial load to the specimen, the pressure chamber lid may be fitted with a piston passing through the top.

6.2 *Specimen Platens*—Both the active-end and passive-end platens shall be constructed of noncorrosive material having a modulus at least ten times the modulus of the material to be tested. Each platen shall have a circular cross section and a plane surface of contact with the specimen, except that the plane surface of contact shall be roughened to provide for more efficient coupling with the ends of the specimen. Roughening and flow of fluids into or from the specimen shall be accomplished by rigidly fastening porous disks to the platens. The diameter of platens shall be equal to or greater than the diameter of the specimen. The construction of the platens shall be such that their stiffness is at least ten times the stiffness of the specimen.

6.2.1 The active-end platen may have a portion of the excitation device, transducers, springs, and dashpots connected to it. The transducers and moving portions of the excitation device must be connected to the platen in such a fashion that they are to be considered part of the platen, be counterbalanced to maintain rotational symmetry, and have the same motion as the platen for the full range of frequencies to be encountered when testing soils.

6.2.2 The theoretical model used for the resonant-column system represents the active-end platen, with all attachments, as a rigid mass that is attached to the specimen; this mass may also have massless springs and dashpots attached to it as shown in Fig. 1. If springs are used, the excitation device and active-end platen (without the specimen in place) form a single-degree-of-freedom system having an undamped natural frequency, f_a.

6.2.3 The passive-end platen must be rigidly fixed. It may be assumed to be rigidly fixed when the inertia of it and the mass(es) attached to it are at least 500 times the inertia of the active-end platen (**1**)[3].

6.2.4 For Device Type 2, a torque transducer is placed between the passive end platen and the rigidly fixed base. The torque transducer even though relatively stiff in torsion (see 6.4), must allow for some small rotation of the passive end platen in order to register the transmitted torque. The inertia of the passive end platen system, J_p must include the inertia of the sensing head of the torque transducer which is rigidly fastened to it. With no specimen in place, the passive end platen system inertia, J_p, along with the stiffness, k_p, and damping coefficient, c_p, of the torque transducer constitute a single-degree-of-freedom system which are accounted for in Eq A1.3.

6.3 *Vibration Excitation Device (torque motor)*—This shall be a device capable of applying a sinusoidal torsional vibration to the active-end platen to which the moving parts of the device are rigidly coupled. The frequency of excitation shall be continuously variable and have a range that typically includes 10 Hz to 1 kHz. For Device Type 1 where the torque is measured at the active end, the excitation device shall have a means of measuring the torque applied to the excitation device that has at least 5 % accuracy of full-scale output. If an electromagnetic excitation device is used, the voltage drop across a fixed, temperature-and-frequency-stable power resistor in series with the excitation device is proportional to applied torque (Note 2). For Device Type 2, the torque is measured at the passive end with a torque transducer, see 6.4.

NOTE 2—Calibrations at more than one frequency may be needed when testing frequencies vary over a wide range. Use of several calibration rods with differing torsional stiffness may be needed.

6.4 *Passive End Torque Transducer*—This torque transducer for Device Type 2 must be waterproof and insensitive to ambient pressure and temperature changes for the expected values. It may be a transducer that also measures axial force. The torque transducer must have a torque capacity of at least twice the maximum torque capability of the vibration excitation device, a linearity of ±0.5 % of full-scale output, hysteresis less than ±0.1 % of full-scale output, and repeatability better than ±0.5 % of full-scale output. If the transducer is used to measure axial force, the specifications must be similar to those for torque. The transducer must be rigidly connected to the chamber base and the sensing head of the torque transducer shall be rigidly connected to the passive end platen.

6.5 *Sine Wave Generator*—The sine wave generator is an electronic instrument capable of producing a sinusoidal current with a means of adjusting the frequency over the entire range of operating frequencies anticipated. This instrument shall provide sufficient power to produce the desired vibration amplitude, or its output may be electronically amplified to provide sufficient power.

6.6 *Vibration-Measuring Devices and Readout Instruments*—These devices and instruments shall be calibrated with an accuracy of 5 % and must be traceable to a government standards agency. The vibration-measuring devices shall be acceleration, velocity, or displacement transducers that can be attached to and become a part of the active-end platen. The transducer(s) shall be mounted to produce a calibrated electrical output that is proportional to the rotational acceleration, velocity, or displacement. If the rotation is measured using linear motion measurement transducers acting at a radius from the axis of rotation, two transducers must be used with both at the same radius from the axis of rotation but connected diagonally across the axis of rotation and they must be wired such that rotational motion is additive and lateral translation is subtracted (Note 3). The readout instruments must have a frequency resolution of at least 0.1 Hz. It also is necessary to have an electronic device for establishing the phase difference between the applied and/or measured torque and resulting rotational motion.

NOTE 3—The use of two linear motion transducers used to measure rotation will minimize resonances in the bending modes from being confused with those for the torsional modes.

6.6.1 For Device Type 1, two types of tests are commonly used: 1) steady-state vibration where applied frequency is varied. (A dual channel digital oscilloscope, readout device, or spectrum analyzer may be used.) and 2) transient vibration starting from a steady-state vibration at the resonant frequency and excitation is then cut off and the decay of vibratory motion is measured with time (a digital *x-y*-time oscilloscope may be used for this purpose). The electronic measuring device used must have amplifiers with sufficient gain to observe the torque motor input and motion transducer outputs over the entire range of frequencies anticipated. For measurement of damping by the free-vibration method, and for calibration of the apparatus damping, the readout instrument shall be capable of

[3] The boldface numbers in parentheses refer to a list of references at the end of this standard.

recording the decay of free vibration with appropriate response time. A digital *x-y*-time oscilloscope may be used for this purpose.

6.6.2 For Device Type 2, a dual channel readout device or a spectrum analyzer must be used to measure the magnitude and phase (or real and imaginary) components of the measured rotation of the active end relative to the applied torque at the passive end.

6.7 *Support for Vibration Excitation Device*—It may be necessary to support all or a portion of the weight of the active-end platen and excitation device to prevent excessive axial stress or compressive failure of the specimen. This support may be provided by a spring, counterbalance weights, or pneumatic device if the supporting system does not prevent axial movement of the active-end platen and if it does not alter the vibration characteristics of the excitation device.

6.8 *Temporary Platen Support Device*—Temporary support of the active-end platen may be any clamping device that can be used to support the platen during attachment of vibration excitation device to prevent specimen disturbance during apparatus assembly. This device is to be removed prior to the application of vibration.

6.9 *Specimen Dimension-Measuring Devices*—Dimension-measuring devices are needed to measure portions of the apparatus during calibration and specimen diameter and length. Any suitable device may be used to make these measurements except that the device(s) used to measure the length and diameter of the specimen must not deform or otherwise affect the specimen. Specially designed perimeter tapes[4] that measure circumference but read out in diameter are preferred for measuring specimen diameters. Measurement accuracies are specified in 7.2.

6.10 *Balances*—Devices for determining the mass of the soil specimens as well as portions of the device during calibration. All measurements of mass shall be accurate to 0.1 %. (Guide D4753)

6.11 *Specimen Preparation and Triaxial Equipment*—These methods cover specimen preparation and procedures related to the vibration of the specimen and do not cover the application and control of static axial and lateral stresses. Any or all of the apparatus described in Test Method D2166/D2166M, D2850, or D4767 may be used for specimen preparation and application of static axial and lateral stresses. Additional apparatus may be used for these purposes as needed.

6.12 *Auxiliary Equipment*—The auxiliary equipment consists of specimen trimming and carving instruments, a membrane expander, remolding apparatus, and moisture content containers as required.

[4] The sole source of supply of the apparatus known to the committee at this time is **PI Tape**, Box 398, Lemon Grove, CA 92045 (http://www.pitape.com). If you are aware of alternative suppliers, please provide this information to ASTM International Headquarters. Your comments will receive careful consideration at a meeting of the responsible technical committee,[1] which you may attend.

7. Test Specimen

7.1 *General*—These methods are limited to the special specimen preparation procedures related to the vibration and resonant-column technique. Since the resonant-column test may be conducted in conjunction with controlled static axial and lateral stresses, the provisions for preparation of specimens in Test Method D2166/D2166M, D2850, or D4767 may be applicable or may be used as a guide in connection with other methods of application and control of static axial and lateral stresses.

7.2 *Specimen Size Limitations*—Specimens shall be of uniform circular cross section with ends perpendicular to the axis of the specimen. Specimens shall have a minimum diameter of 33 mm (1.3 in.). The largest particle contained within the test specimen shall be one sixth of the specimen diameter. If, after completion of a test, it is found that larger particles than permitted are present, indicate this information in the report of test data under "Remarks." The length-to-diameter ratio shall be not less than 2 nor more than 7 except that, when a static axial stress greater than the lateral stress is applied to the specimen, the ratio of length to diameter shall be between 2 and 3. Take diameter measurements to the nearest 0.25 mm (0.01 in.), at the third points along the specimen length and average them. Take height measurements, to the nearest 0.25 mm (0.01 in.), at four quadrants and average them. For determination of moisture content (Test Method D2216), secure a representative specimen of the cuttings from intact specimens, or of the extra soil for remolded specimens, placing the specimen immediately in a covered container.

7.3 *End Coupling for Torsion*—For torsional motion, complete coupling of the ends of the specimen to the specimen cap and base must be assured. Coupling for torsion may be assumed if the mobilized coefficient of friction between the end platens and the specimen is less than 0.2 for all shear strain amplitudes. The coefficient of friction is approximately given by:

$$\text{Mobilized Coefficient of Friction} = \frac{\gamma G}{\sigma'_a} \quad (1)$$

where:

γ = shear strain amplitude (see Calculations section),
G = shear modulus (see Calculations section), and
σ'_a = effective axial stress.

NOTE 4—The shear strain is not in % for this calculation.

7.3.1 When this criterion is not met, other provisions such as the use of adhesives or other friction increasing measures must be made in order to assure complete coupling (**2**). In such cases, the effectiveness of the coupling provisions shall be evaluated by testing two specimens of the same material but of different length. The lengths of these specimens shall differ by at least a factor of 1.5. The provisions for end coupling may be considered satisfactory if the values of the shear modulus for these two specimens of different length do not differ by more than 10 %.

8. Apparatus Properties (see Note 5)

NOTE 5—Practice D3740 provides information on calibration intervals, records, and quality assurance.

8.1 *Motion Transducers*—Motion transducers shall be calibrated with an independent method to ensure calibration accuracy within 5 % and must be traceable to a government standards agency.

8.1.1 *Rotational Motion Transducer*—The rotational motion at the free end of the soil specimen is normally measured using linear motion transducer(s) mounted at a radial distance r_t from the axis of rotation. Linear motion transducers that are sensitive to acceleration, velocity or displacement may be used. Rotational measuring transducers are acceptable as well. (See 6.6.)

8.1.1.1 The rotation transducer sensitivity S_θ in terms of millivolts/radian is computed as follows:

For an accelerometer transducer with sensitivity S_a [mV/g] the rotation transducer sensitivity at frequency f [Hz] is:

$$S_\theta = S_a r_t (2\ \pi\ f)^2 (1 / 9.81) \tag{2}$$

For a velocity transducer with sensitivity S_v [mV/(m/s)] the rotation transducer sensitivity is:

$$S_\theta = S_v r_t (2\ \pi\ f) \tag{3}$$

For a displacement transducer with sensitivity S_d [mV/m] the rotation transducer sensitivity is:

$$S_\theta = S_d r_t \tag{4}$$

Rotation of the top of the specimen is given by:

$$\theta[\text{rad}] = \frac{RTrdg[\text{mV}]}{S_\theta\left[\frac{\text{mV}}{\text{rad}}\right]} \tag{5}$$

where *RTrdg* is the output of the rotation transducer.

8.2 *Active-End Rotational Inertia (only needed for Device Type 1)*—The rotational inertia, J_a, of the active-end platen shall be determined with all transducers and rigid attachments, including attached portions of the vibration excitation device, securely in place. The rotational inertia of the concentric solid cylindrical components of the active-end platen and its attachments is computed from:

$$(J_a)_1 = \frac{1}{8}\sum_{i=1}^{n} M_i d_i^2 \tag{6}$$

where:

M_i = mass of i^{th} solid cylindrical component,
d_i = diameter of i^{th} solid cylindrical component, and
n = number of solid cylindrical components.

Transducers and other masses attached to this platen can be accounted for by:

$$(J_a)_2 = \sum_{i=1}^{n} (J_i + M_i\ r_i^2) \tag{7}$$

where:

J_i = rotational inertia of the i^{th} component,
M_i = mass of i^{th} component,
r_i = distance from the platen axis to center of mass for i^{th} component, and
n = number of components attached to active-end platen and not covered in determination of $(J_a)_1$.

The total rotational inertia for the active end is given by:

$$J_a = (J_a)_1 + (J_a)_2 \tag{8}$$

8.2.1 Acceptable alternate procedures for determining J_a are provided in A2.1.

8.3 *Apparatus Resonant Frequencies, Spring Constants, and Damping Constants (only needed for Device Type 1)*—(See Note 6) Apparatus resonant frequencies and spring constants are defined only for Device Type 1 that has springs attached to the active-end platen system. To determine the resonant frequencies, set up the apparatus complete with active-end platen and O-rings used to seal the membranes, but with no specimen. Vibrate at low amplitude and adjust the frequency of vibration until the input torque is in phase with the velocity of the active-end platen system. This apparatus resonant frequency is f_a. The apparatus spring constant, k_a, is calculated from:

$$k_a = (2\ \pi\ f_a)^2 J_a \tag{9}$$

where J_a is defined in the previous subsection.

NOTE 6—Device Type 2 apparatus may or may not have springs and dashpots attached to the active end platen but by Eq A1.3, these and the active end platen inertia do not affect the determination of shear modulus and damping of the soil.

8.3.1 Apparatus Damping Coefficient for Device Type 1 apparatus without springs attached to the active end platen. Device Type 1 without springs may still have a damping constant to account for back EMF, aerodynamic drag, vibration of wires attached to the platen, and eddy currents. To measure the damping constants for the apparatus, attach the same masses as used for the determination of apparatus resonant frequencies. For apparatus without springs attached to the active-end platen, insert the calibration rod described in the previous subsection. Vibrate the system at the resonant frequency and measure the torque and rotational motion. The apparatus damping coefficient is given by:

$$c_a = \frac{\tau_{appl}}{\theta\omega} = \frac{\tau_{appl}}{\frac{d\theta}{dt}} = \frac{\tau_{appl}\omega}{\frac{d^2\theta}{dt^2}} \tag{10}$$

where:

τ_{appl} = amplitude of applied torque,
θ = amplitude of rotation,
$\frac{d\theta}{dt}$ = amplitude of rotational velocity,
$\frac{d^2\theta}{dt^2}$ = amplitude of rotational acceleration, and
ω = resonant circular frequency of the system at calibration (=$2\pi f$).

8.3.2 An acceptable alternate method for calculating the apparatus damping coefficient, c_a is given in A2.2. Reference (**3**) provides a convenient method for determining both J_a and c_a that makes use of the program given in Appendix X1.

8.4 *Torque Motor Torque/Current Characteristics (only needed for Device Type 1)*—For Device Type 1 apparatus without springs attached to the active-end platen, insert the calibration rod as described earlier. For Device Type 1 apparatus with springs attached, set up the apparatus complete with active-end platen and O-rings but no specimen. For either setup, determine the resonant frequency of this single-degree-of-freedom system consisting of the active-end platen and apparatus spring (or calibration rod) by use of the same procedure as described later in the procedures section. Then set the frequency to 0.707 times the resonant frequency and apply torque so that the vibration transducer output to the readout device has a signal of at least ten times the signal due to ambient vibrations and electrical noise when no torque is applied. Read and record the output of the vibration transducer and the current input to the torque generating instrument (torque motor). Next, set the frequency to 1.414 times the system resonant frequency and obtain the readings similar to those at 0.707 times the resonant frequency. Calculate C_1 and C_2 from:

$$C_1 = \frac{\theta_1}{2CR_1} \tag{11}$$

$$C_2 = \frac{\theta_2}{CR_2}$$

where:

θ_1 = active-end rotation at 0.707 times resonant frequency (Note 7),
CR_1 = torque motor input (amps) at 0.707 times resonant frequency (Note 8),
θ_2 = active-end transducer output at 1.414 times resonant frequency (Note 7), and
CR_2 = torque motor input (amps) at 1.414 times resonant frequency (Note 8).

NOTE 7—θ_1 and θ_2 will be functions of frequency for velocity and acceleration measuring transducers (see 8.1).

NOTE 8—If a current-measuring instrument is used, the units will be amperes. Alternatively, voltage drop across a fixed resistance may also be measured and the units will then be volts.

By use of C_1 and C_2, the torque motor rating, *TMR*, is obtained from:

$$TMR = 0.5k(C_1 + C_2) \tag{12}$$

where:

k = apparatus spring constant, k_a (or for apparatus without springs, the calibrating rod spring constant, k_{rod}).

The torque applied to the top platen by the torque generator is given by:

$$\tau_{appl} = TMR \cdot T_{rdg} \tag{13}$$

where:

T_{rdg} = input amps to the torque motor
TMR = torque motor rating from Eq 12.

8.5 *Passive End Inertia and Torque Transducer Calibrations (only needed for Device Type 2):*

8.5.1 A torque transducer generally consists of a metallic case containing a “spring” instrumented to measure strain where the strain is proportional to the applied torque. The torque is applied to the spring through a sensing head protruding from the transducer case. The sensing head must be rigidly connected to the passive end platen and provide the basis for the passive end rotational inertia:

$$J_p = J_{passive\ platen} + J_{sens\ head} \tag{14}$$

where:

$J_{passive\ platen}$ = calculated using Eq 6-8, and
$J_{sens\ head}$ = frequently is provided by the transducer manufacturer.

8.5.2 Alternative methods provided in A2.3.

8.5.3 The torque transducer sensitivity is given by the manufacturer and must be traceable to a government standards agency. The torque measured by the torque transducer is calculated from:

$$\tau_{TT} = \frac{TT_{rdg}}{TT_{sens}} \tag{15}$$

where:

TT_{sens} = Torque Transducer sensitivity typically in units of mV/(N-m)
TT_{rdg} = Voltage reading (mV) for the torque transducer.

9. Procedure

9.1 *Test Setup*—The exact procedure to be followed during test setup will depend on the apparatus and electronic equipment used and on methods used for application, measurement, and control of the static axial and lateral stresses. However, the specimen shall be placed in the apparatus by procedures that will minimize the disturbance of the specimen. Particular care must be exercised when attaching the end platens to the specimen and when attaching the vibration excitation device to the platens. A temporary support as discussed earlier may be needed. For cases where isotropic static stresses are to be applied to a membrane-enclosed specimen, liquid- or air-confining media may be used for dry or partially saturated specimens. For tests where complete saturation is important, a liquid-confining medium shall be used. Where the vibration excitation device is located within the pressure chamber, an air-liquid interface is acceptable if the liquid covers the entire membrane that encloses the specimen.

9.2 *Electronic Equipment*—Turn off the power supplied to the torque motor. Connect the torque motor to the sine wave generator (with amplifier, if required). Connect the vibration transducers to the readout instruments. Gradually apply power to the torque motor and adjust the readout instruments according to the instruction manuals for these instruments.

9.3 *Measurements:*

9.3.1 *Device Type 1:*

9.3.1.1 *Measurement of Resonant Frequency*—The motion of the active-end platen in conjunction with the applied torque is used to establish resonance. Resonance is defined as the lowest frequency where the torque is 90 degrees out-of-phase with the rotational acceleration or displacement. This phase relationship can be detected by observing the Lissajous figure on an oscilloscope with the torque input signal and rotational acceleration or displacement plotted as *x-y*. (Note 9) At the 90 degree phase relationship the figure will be an ellipse with its axes vertical and horizontal. If a velocity transducer is used for rotational measurement, the system resonance occurs when the

Lissajous figure forms a straight, sloping line. It is recommended that the frequency be measured with a digital electronic frequency meter and be recorded to at least three significant figures.

9.3.1.2 The determination of the lowest resonant frequency can be done by setting the torque excitation frequency (for example, 10 Hz) and power to as low a value as practical. Then increase the frequency of excitation until the system resonant frequency is obtained.

NOTE 9—The phase relationship between two signals may also be computed by measurement of the time difference between zero crossings of the two signals divided by the period of the oscillations ($period = \frac{1}{frequency}$) multiplied by 360 gives the phase in degrees. If the signals are not clean sine waves, then a spectral analysis will have to be performed to get accurate values for magnitude and phase (or real and imaginary components) of the rotation/torque ratio. The magnitude of the rotation/torque ratio multiplied by the cosine of the phase gives the real component of the rotation/torque ratio and the same ratio multiplied by the sine of the phase gives the imaginary component of the rotation/torque ratio.

9.3.1.3 *Measurement of Strain*—The strain amplitude measurements shall be made only at the system resonant frequencies. Thus, for a given torque, the vibration motion transducer outputs recorded at the system resonant frequency give sufficient information to calculate strain amplitude. To increase or decrease strain amplitude, the applied torque must be increased or decreased. After making a change in torque, the procedure of 9.3.1.1 must be followed to establish the corresponding system resonant frequency before the rotation transducer output can be used to establish the new shear strain amplitude value.

9.3.1.4 *Measurement of System Damping*—Associated with each shear strain amplitude and system resonant frequency is a value of damping. Two methods are available for measuring system damping: the steady-state vibration method and the amplitude decay method. Both methods should give similar results. The steady-state method is easier and quicker. It is generally always used and the amplitude decay method is used for occasional spot-checking. For the steady-state method, the active-end transducer output and the applied torque must be measured at each resonant frequency. The calculations are outlined in the following section. For the free-vibration method, with the system vibrating at the system resonant frequency, cut the power to the vibration excitation device and record the output of the rotation transducer used in establishing resonance as a function of time. The shut-off mechanism must create an open circuit with the vibration excitation device and cannot be done by switching off the power to amplifier. Without an open circuit, damping will be induced by current flow in the circuit. This gives the decay curve for free vibration. The calculations for damping are outlined in the following section.

9.3.2 *Device Type 2:*

9.3.2.1 *Measurement of Resonant Frequency*—This is the lowest frequency at which the active end rotation is a maximum. In addition to measuring the frequency, magnitude of motion and magnitude of torque, the phase between the motion at the active end and the torque at the passive end must be determined (see Note 9).

9.3.2.2 *Measurement of Strain Amplitude*—The strain amplitude measurements shall be made only at the system resonant frequencies. Thus, for a given torque, the vibration motion transducer outputs recorded at the resonant frequency give sufficient information to calculate strain amplitude. To increase or decrease strain amplitude, the applied torque must be increased or decreased. After making a change in torque, the procedure of 9.3.2.1 must be followed to establish the corresponding resonant frequency before the rotation transducer output can be used to establish the new shear strain amplitude value.

9.3.2.3 *Measurement of System Damping*—Damping is determined from steady-state measurements of torque measured at the base of the specimen (passive end), amplitude of motion of the active end and the phase difference between them as described in the next section.

10. Calculation

10.1 *General*—Calculations require the apparatus calibration factors and the physical dimensions and mass of the specimen at the time resonant measurements are made. In addition, for each static axial and lateral stress condition, one data set shall be measured for each vibration strain amplitude. A data set consists of: duration of vibration (this time can be used to calculate the number of vibration cycles), system resonant frequency, active-end transducer output for both type devices. For Device Type 1 additionally, the reading associated with the applied torque, and if the amplitude decay method of measuring damping is also going to be used, the free-vibration amplitude decay curve. For Device Type 2, it is necessary to measure the torque as well as the phase between the torque transducer output and the motion at the active end of the specimen (Note 9).

10.1.1 The calculations outlined in this section may all be made by computer programs. For Device Type 1, a program for making the calculations is provided in Appendix X1. For Device Type 2, the program is given in Appendix X2. Other programs may be used to make a portion or all of the calculations as long as they provide identical results. The units for the symbols in this section are given in Annex A3.

10.2 *Soil Mass Density*—The soil mass density, ρ, is given by:

$$\rho = \frac{M}{V} \tag{16}$$

where:

M = total mass of specimen, and
V = volume of specimen.

10.3 *Specimen Rotational Inertia*—The specimen rotational inertia about the axis of rotation is given by:

$$J = \frac{Md^2}{8} \tag{17}$$

where d = diameter of specimen.

10.4 *Active-End Inertia Factors:*

10.4.1 The active-end inertia factor, T_a, is only needed for Device Type 1 and is given by:

$$T_a = \frac{J_a}{J}\left[1 - \left(\frac{f_a}{f_r}\right)^2\right] \tag{18}$$

where:

J_a = rotational inertia of active-end platen system as calculated earlier,
J = specimen rotational inertia as calculated earlier,
f_a = apparatus resonant frequency (for apparatus without springs attached to the active-end platen, this factor is zero), and
f_r = system resonant frequency.

10.5 *Apparatus Damping Factors:*

10.5.1 The apparatus damping factor, for Device Type 1 is calculated from:

$$ADF_a = \frac{c_a}{2\pi f_r J} \tag{19}$$

where c_a = apparatus damping coefficient as described by Eq 10 or A2.2.

10.6 *Modified Magnification Factor:*

10.6.1 The measured modified magnification factor is used in calculating both modulus and damping. For Device Type 1, it is calculated from:

$$(MMF_{meas})_{DT1} = J\omega^2\left[\mathrm{Re}\left(\frac{\theta_a}{\tau_{appl}}\right) + i\mathrm{Im}\left(\frac{\theta_a}{\tau_{appl}}\right)\right] \tag{20}$$

where:

θ_a = rotational motion at the active end,
τ_{appl} = torque applied to the active end, and
ω = resonant frequency as defined in 9.3.1.1 or 9.3.1.2.

At resonance where the phase is –90 degrees, the real portion becomes zero.

10.6.2 For Device Type 2, the measured magnification factor is given by:

$$(MMF_{meas})_{DT2} = J\omega^2\left[\mathrm{Re}\left(\frac{\theta_a}{\tau_{TT}}\right) + i\mathrm{Im}\left(\frac{\theta_a}{\tau_{TT}}\right)\right] \tag{21}$$

where:

θ_a = rotational motion at the active end,
τ_{TT} = torque measured by the torque transducer at the passive end, and
ω = resonant frequency as described in 9.3.2.1.

10.7 *Shear Modulus and Damping:*

10.7.1 *Governing Equation*—For Device Type 1, substituting Eq 17 and 18 into Eq A1.2 with some rearrangement gives Eq A1.4, the Dimensionless Specimen Stiffness (DSS):

$$DSS^*_{meas} = \left[\frac{1}{(MMF_{meas})_{DT1}} + T_a - ADF_a\right] \tag{22}$$

where:

MMF_{meas} = calculated value from Eq 20,
T_a = active end inertia factor from Eq 18, and
ADF_a = apparatus damping factor calculated from Eq 19.

10.7.1.1 The calculated Dimensionless Specimen Stiffness from Eq A1.5 is given by:

$$DSS^*_{calc} = \frac{1}{\lambda^*\tan(\lambda^*)} \tag{23}$$

where λ^* is defined by Eq A1.4.

10.7.1.2 The dimensionless frequency, λ* is determined by setting $DSS_{meas}{}^*$ from Eq 22 equal to $DSS_{calc}{}^*$ from Eq 23 using an optimization routine. The computer program in Appendix X1, which is written in Excel, solves for λ* by comparing Eq 23 the results with Eq 22.

10.7.1.3 For Device Type 2, multiplying Eq A1.3 by $\omega^2 J$ and with the assumption that $\omega c_p << k_p$ gives:

$$(MMF_{calc})_{DT2} = \frac{J}{J_p}\left(\frac{\omega}{\omega_p}\right)^2\cos\lambda^* + \left[1 - \left(\frac{\omega}{\omega_p}\right)^2\right]\lambda^*\sin\lambda^* \tag{24}$$

where λ* is defined by Eq A1.4 and

$$\omega_p = \sqrt{\frac{k_p}{J_p}} \tag{25}$$

10.7.2 *Dimensionless Frequency Factor*—The dimensionless frequency factor, λ^*, is complex having both a real component, λ_{Re}, and an imaginary component, λ_{Im}. It is used in calculating modulus and damping by solving Eq 22 and Eq 23 for Device Type 1 or Eq 21 and Eq 24 for Device Type 2.

10.7.3 For Device Type 2, $(MMF_{calc})_{DT2}$ by Eq 24 is a function of J_p / J, ω / ω_p, and λ^*. The computer program in Appendix X2, which is written in Excel, compares calculated values of $(MMF_{calc})_{DT2}$ using Eq 24 with measured values $(MMF_{meas})_{DT2}$ by Eq 21 and provides values of λ^*.

10.7.4 The dimensionless modulus factor is defined as:

$$F_a = \frac{\lambda_{Re}^2 - \lambda_{Im}^2}{(\lambda_{Re}^2 + \lambda_{Im}^2)^2} \tag{26}$$

where the subscripts "Re" and "Im" refer to the real and imaginary components of λ^*. F_a also is given by the Excel program in Appendix X1 for Device Type 1 and the program in Appendix X2 for Device Type 2.

10.7.5 *Shear Modulus*—The shear modulus for both Device Type 1 and Device Type 2 is calculated from:

$$G = \rho(\omega\ L)^2 F_a \tag{27}$$

where:

ρ = density of the soil specimen,
L = specimen length,
ω_r = system resonant circular frequency = $2\pi f_r$, and
F_a = dimensionless modulus factor from Eq 26 for both Device Type 1 and Device Type 2.

10.8 *Damping Ratio:*

10.8.1 *Damping Ratio from Steady State Vibration*—The damping ratio for both devices is calculated from:

$$D = \frac{-\lambda_{Re}\lambda_{Im}}{\lambda_{Re}^2 - \lambda_{Im}^2} \tag{28}$$

where D is in decimal form.

10.8.2 *Damping Ratio from Free Vibration*—This method only applies to Device Type 1 and with no springs attached to the active end platen. The system must be operating at the resonant frequency. The transducer that is used to determine rotation amplitude must be used to obtain the amplitude decay curve. Determine the system logarithmic decrement from:

$$\delta = \left(\frac{1}{n}\right)\ln\left(\frac{A_1}{A_{n+1}}\right) \tag{29}$$

where:

A_1 = amplitude of vibration for first cycle after power is cut off,

A_{n+1} = amplitude of vibration for $(n + 1)^{th}$ cycle of free vibration, and

n = number of free vibration cycles which must be 10 or less.

10.8.2.1 System damping by amplitude decay is given by:

$$D_{system} = \frac{\delta}{\sqrt{\delta^2 + (2\pi)^2}} \tag{30}$$

Damping in the specimen then is calculated from:

$$D_s = \frac{1}{2\pi}\delta - \frac{c_a}{2\sqrt{kJ_a}} \tag{31}$$

where

$$k = k_a + \frac{G\pi d^4}{32L} \tag{32}$$

Free vibration solutions do not apply to Device Type 2.

10.9 *Strain Amplitude:*

10.9.1 The average shear strain amplitude, γ_{avg}, shall be calculated from:

$$\gamma_{avg}(\%) = \frac{r_{avg}}{L}(\theta_a - \theta_p)\ 100\% \tag{33}$$

where:

θ_a = magnitude of the rotational motion at the active end,

θ_p = τ_{TT} / k_p For Device Type 2 (For Device Type 1, $\theta_p = 0$),

L = specimen length,

r_{avg} = 0.4d is the default value (values between 0.33d to 0.40d may be used if documented in the report, and

d = specimen diameter.

11. Report: Test Data Sheet(s)/Form(s)

11.1 The methodology used to specify how data are recorded on the test data sheet(s)/form(s), as given below, is covered in Practice D6026, referenced in 1.4.

11.2 Record as a minimum the following general information (data):

11.2.1 Date of test, operator name, location of test.

11.3 Record as a minimum the following Apparatus Characteristics data for Device Type 1:

11.3.1 Apparatus name, model number, and serial number;

11.3.2 Active-end rotational inertia (J_a);

11.3.3 Apparatus resonant frequency (f_a) for apparatus with a spring attached to the top platen;

11.3.4 Apparatus damping coefficient (c_a);

11.3.5 The torque motor rating (*TMR*);

11.3.6 The motion transducer calibration factor (S_θ).

11.4 Record as a minimum the following Apparatus Characteristics data for Device Type 2:

11.4.1 Apparatus name, model number, and serial number;

11.4.2 The torque transducer stiffness (k_p);

11.4.3 The torque transducer sensitivity (TT_{sens});

11.4.4 The passive end inertia (J_p);

11.4.5 The motion transducer calibration factor (S_θ).

11.5 Record as a minimum the following Specimen Characteristics data:

11.5.1 A visual description and origin of the soil shall be given, including name, group symbol, and whether intact or remolded.

11.5.2 Initial and final specimen mass, diameter, length, void ratio, water content, and degree of saturation.

11.5.3 Specimen preparation procedures and test setup procedures shall be outlined.

11.6 Record as a minimum the following Static Test Conditions data:

11.6.1 A complete description of the static axial and lateral stress conditions shall be given, including total stresses and pore water pressures, drainage conditions, and the procedures used to measure applied stresses, pore pressures, length change, and volume change.

11.7 Record as a minimum the following for Each Data Set:

11.7.1 Approximate time of vibration at this strain amplitude,

11.7.2 Cell pressure, backpressure or pore pressure, axial stress,

11.7.3 Specimen length, volume, and density,

11.7.4 Radius used for calculating average shear strain if different from 0.4d, and

11.7.5 System resonant frequency, strain amplitude, shear modulus, and damping ratio.

12. Precision and Bias

12.1 *Precision*—Test data on precision are not presented due to the nature of the soil or rock, or both materials tested by this standard. It is either not feasible or too costly at this time to have ten or more laboratories participate in a round-robin testing program. In addition, it is either not feasible or too costly to produce multiple specimens that have uniform physical properties. Any variation observed in the data is just as likely to be due to specimen variation as to operator or laboratory testing variation.

12.2 Subcommittee D18.09 is seeking any pertinent data from users of these test methods that might be used to make a limited statement on precision.

12.3 *Bias*—There is no accepted reference value for these test methods, therefore, bias cannot be determined.

13. Keywords

13.1 amplitude; confining pressure; damping; dynamic loading; elastic waves; frequency; laboratory tests; nondestructive tests; resonance; shear modulus; shear tests; soils; strain; stress; torsional oscillations; transfer function method; triaxial stress

ANNEXES

(Mandatory Information)

A1. THEORY

INTRODUCTION

The shear modulus shall be defined as the elastic shear modulus of a uniform, linearly viscoelastic (Voigt model) specimen of the same mass density and dimensions as the soil specimen necessary to produce a resonant column having the measured system resonant frequency and response due to a given vibratory torque input. The specimen properties can be characterized by a specimen stiffness transfer function matrix (**4**, **5**, **6**, **7**) which greatly simplifies the solution to the system. The stress-strain relation for a steady-state vibration in the resonant column is a hysteresis loop. This modulus, *G*, will correspond to the slope of a line through the end points of the hysteresis loop. The section on calculations provides for computation of shear moduli from the measured system torsional resonant frequencies. The energy dissipated by the system is a measure of the damping of the soil. Damping will be described by the shear damping ratio, *D*, which is analogous to the critical viscous damping ratio, c/c_c, for a single-degree-of-freedom system (**3**).

A1.1 Values of damping determined in this way will correspond to the area of the stress-strain hysteresis loop divided by four times the elastic strain energy stored in the specimen at maximum strain. Methods for determining damping ratio are prescribed later. In viscoelastic theory, it is common to use complex moduli to express both modulus and damping. The complex shear modulus is given by:

$$G^* = G(1 + i\,2\,D) \quad \text{(A1.1)}$$

where $i = \sqrt{-1}$

A1.2 Equilibrium Equations

A1.2.1 The equation describing the rotation of the active-end platen for a given applied torque to the active end (Device Type 1) is given by (**5**, **7**):

$$\frac{\theta_a}{\tau_{appl}} = \frac{1}{\dfrac{\omega^2 J}{\lambda^* \tan\lambda^*} - \omega^2 J_a + i\omega c_a + k_a} \quad \text{(A1.2)}$$

where:

- θ_a = the rotation at the active (top) end,
- τ_{appl} = the torque applied to the active end,
- ω = the excitation circular frequency (= $2\pi f$),
- J = the polar mass moment of inertia of the specimen,
- J_a = the total polar mass moment of inertia of the top platen system including the apparatus inertial contribution calculated from calibration,
- c_a = the apparatus damping coefficient for the active end,
- k_a = the apparatus stiffness, and
- λ^* = the complex dimensionless frequency.

A1.2.2 The equation describing the rotation at the active end platen for torque measured at the passive end platen (Device Type 2) is given by (**5**, **7**):

$$\frac{\theta_a}{\tau_{TT}} = \frac{\dfrac{\omega^2 J}{\lambda^* \tan\lambda^*} - \omega^2 J_p + k_p}{k_p \dfrac{\omega^2 J}{\lambda^* \sin\lambda^*}} \quad \text{(A1.3)}$$

where:

- θ_a = the rotation at the active (top) end,
- τ_{TT} = the torque measured by the torque transducer at the passive end,
- J = the polar mass moment of inertia of the specimen,
- J_p = the total polar mass moment of inertia of the bottom platen system including the torque transducer inertial contribution calculated from calibration,
- k_p = the torque transducer stiffness, and
- λ^* = the complex dimensionless frequency.

A1.2.3 For Device Type 1, the measured dimensionless specimen stiffness (DSS_{meas}), removes the apparatus active end platen inertia, apparatus spring stiffness, and apparatus damping from the measured MMF_{meas} in Eq 20.

$$DSS^*_{meas} = \left[\frac{1}{(MMF_{meas})_{DT1}} + T_a - ADF_a\right] \quad \text{(A1.4)}$$

A1.2.4 The calculated value of $DSS_{calc}{}^*$ is given by Eq A1.5. It is only a function of λ^* which is used for determining the modulus and damping in the specimen.

$$DSS^*_{calc} = \frac{1}{\lambda^* \tan(\lambda^*)} \quad \text{(A1.5)}$$

The determination of λ^* involves finding values of λ^* such that the difference between $DSS_{calc}{}^*$ and $DSS_{meas}{}^*$ is a minimum. This is done by using an optimization routine such as in the Excel program in Appendix X1.

A1.2.5 The complex dimensionless frequency, λ^*, is a parameter that characterizes the properties of the specimen as defined by (**5**, **6**, **7**):

$$\lambda^* = \frac{\omega L}{\sqrt{\dfrac{G[1 + i\,2\,D]}{\rho}}} \quad \text{(A1.6)}$$

where:

- G = shear modulus of the specimen,