INTERNATIONAL STANDARD

ISO 20501

First edition
2003-12-01

Fine ceramics (advanced ceramics, advanced technical ceramics) — Weibull statistics for strength data

Céramiques techniques — Statistiques Weibull des données de résistance

Reference number
ISO 20501:2003(E)

ISO copyright office
Case postale 56 • CH-1211 Geneva 20
Tel. + 41 22 749 01 11
Fax + 41 22 749 09 47
E-mail copyright@iso.org
Web www.iso.org

Published in Switzerland

Contents

Page

Foreword iv
1 Scope 1
2 Terms and definitions 1
2.1 Defect populations 1
2.2 Mechanical testing 2
2.3 Statistical terms 3
2.4 Weibull distributions 4
3 Symbols 5
4 Significance and use 6
5 Method A: maximum likelihood parameter estimators for single flaw populations 7
5.1 General 7
5.2 Censored data 7
5.3 Likelihood functions 7
5.4 Bias correction 8
5.5 Confidence intervals 9
6 Method B: maximum likelihood parameter estimators for competing flaw populations 11
6.1 General 11
6.2 Censored data 12
6.3 Likelihood functions 12
7 Procedure 13
7.1 Outlying observations 13
7.2 Fractography 13
7.3 Graphical representation 13
8 Test report 16
Annex A (informative) Converting to material-specific strength distribution parameters 17
Annex B (informative) Illustrative examples 19
Annex C (informative) Test specimens with unidentified fracture origin 26
Annex D (informative) Fortran program 29
Bibliography 33

Foreword

ISO (the International Organization for Standardization) is a worldwide federation of national standards bodies (ISO member bodies). The work of preparing International Standards is normally carried out through ISO technical committees. Each member body interested in a subject for which a technical committee has been established has the right to be represented on that committee. International organizations, governmental and non-governmental, in liaison with ISO, also take part in the work. ISO collaborates closely with the International Electrotechnical Commission (IEC) on all matters of electrotechnical standardization.

International Standards are drafted in accordance with the rules given in the ISO/IEC Directives, Part 2.

The main task of technical committees is to prepare International Standards. Draft International Standards adopted by the technical committees are circulated to the member bodies for voting. Publication as an International Standard requires approval by at least 75 % of the member bodies casting a vote.

Attention is drawn to the possibility that some of the elements of this document may be the subject of patent rights. ISO shall not be held responsible for identifying any or all such patent rights.

ISO 20501 was prepared by Technical Committee ISO/TC 206, *Fine ceramics*.

Fine ceramics (advanced ceramics, advanced technical ceramics) — Weibull statistics for strength data

1 Scope

This International Standard covers the reporting of uniaxial strength data and the estimation of probability distribution parameters for advanced ceramics which fail in a brittle fashion. The failure strength of advanced ceramics is treated as a continuous random variable. Typically, a number of test specimens with well-defined geometry are brought to failure under well-defined isothermal loading conditions. The load at which each specimen fails is recorded. The resulting failure stresses are used to obtain parameter estimates associated with the underlying population distribution.

This International Standard is restricted to the assumption that the distribution underlying the failure strengths is the two-parameter Weibull distribution with size scaling. Furthermore, this International Standard is restricted to test specimens (tensile, flexural, pressurized ring, etc.) that are primarily subjected to uniaxial stress states. Subclauses 5.4 and 5.5 outline methods of correcting for bias errors in the estimated Weibull parameters, and to calculate confidence bounds on those estimates from data sets where all failures originate from a single flaw population (i.e., a single failure mode). In samples where failures originate from multiple independent flaw populations (e.g., competing failure modes), the methods outlined in 5.4 and 5.5 for bias correction and confidence bounds are not applicable.

Measurements of the strength at failure are taken for one of two reasons: either for a comparison of the relative quality of two materials, or the prediction of the probability of failure (or alternatively the fracture strength) for a structure of interest. This International Standard permits estimates of the distribution parameters which are needed for either. In addition, this International Standard encourages the integration of mechanical property data and fractographic analysis.

2 Terms and definitions

For the purposes of this document, the following terms and definitions apply.

2.1 Defect populations

2.1.1
censored strength data
strength measurements (i.e., a sample) containing suspended observations such as that produced by multiple competing or concurrent flaw populations

NOTE Consider a sample where fractography clearly established the existence of three concurrent flaw distributions (although this discussion is applicable to a sample with any number of concurrent flaw distributions). The three concurrent flaw distributions are referred to here as distributions A, B, and C. Based on fractographic analyses, each specimen strength is assigned to a flaw distribution that initiated failure. In estimating parameters that characterize the strength distribution associated with flaw distribution A, all specimens (and not just those that failed from type-A flaws) must be incorporated in the analysis to assure efficiency and accuracy of the resulting parameter estimates. The strength of a specimen that failed by a type-B (or type-C) flaw is treated as a *right censored* observation relative to the A flaw distribution. Failure due to a type-B (or type-C) flaw restricts, or censors, the information concerning type-A flaws in a specimen by suspending the test before failure occurs by a type-A flaw [2]. The strength from the most severe type-A flaw in those specimens that failed from type-B (or type-C) flaws is higher than (and thus to the *right* of) the observed strength. However, no information is provided regarding the magnitude of that difference. Censored data analysis techniques incorporated in this International Standard utilize this incomplete information to provide efficient and relatively unbiased estimates of the distribution parameters.

2.1.2
competing failure modes
distinguishably different types of fracture initiation events that result from concurrent (competing) flaw distributions

2.1.3
compound flaw distributions
any form of multiple flaw distribution that is neither pure concurrent, nor pure exclusive

NOTE A simple example is where every specimen contains the flaw distribution A, while some fraction of the specimens also contains a second independent flaw distribution B.

2.1.4
concurrent flaw distributions
a type of multiple flaw distribution in a homogeneous material where every specimen of that material contains representative flaws from each independent flaw population

NOTE Within a given specimen, all flaw populations are then present concurrently and are competing with each other to cause failure. This term is synonymous with “competing flaw distributions”.

2.1.5
exclusive flaw distributions
a type of multiple flaw distribution created by mixing and randomizing specimens from two or more versions of a material where each version contains a different single flaw population

NOTE Thus, each specimen contains flaws exclusively from a single distribution, but the total data set reflects more than one type of strength-controlling flaw. This term is synonymous with “mixture flaw distributions”.

2.1.6
extraneous flaws
strength-controlling flaws observed in some fraction of test specimens that cannot be present in the component being designed

NOTE An example is machining flaws in ground bend specimens that will not be present in as-sintered components of the same material.

2.2 Mechanical testing

2.2.1
effective gauge section
that portion of the test specimen geometry included within the limits of integration (volume, area or edge length) of the Weibull distribution function

NOTE In tensile specimens, the integration may be restricted to the uniformly stressed central gauge section, or it may be extended to include transition and shank regions.

2.2.2
fractography
the analysis and characterization of patterns generated on the fracture surface of a test specimen

NOTE Fractography can be used to determine the nature and location of the critical fracture origin causing catastrophic failure in an advanced ceramic test specimen or component.

2.2.3
proof testing
applying a predetermined load to every test specimen (or component) in a batch or a lot over a short period of time to ascertain if the specimen fails due to a serious strength limiting defect

NOTE This procedure, when applied to all specimens in the sample, removes potentially weak specimens and modifies the statistical characteristics of the surviving samples.

2.3 Statistical terms

2.3.1
confidence interval
interval within which one would expect to find the true population parameter

NOTE Confidence intervals are functionally dependent on the type of estimator utilized and the sample size. The level of expectation is associated with a given confidence level. When confidence bounds are compared to the parameter estimate one can quantify the uncertainty associated with a point estimate of a population parameter.

2.3.2
confidence level
probability that the true population parameter falls within a specified confidence interval

2.3.3
estimator
well-defined function that is dependent on the observations in a sample

NOTE The resulting value for a given sample may be an estimate of a distribution parameter (a point estimate) associated with the underlying population. The arithmetic average of a sample is, e.g., an estimator of the distribution mean.

2.3.4
population
totality of potential observations about which inferences are made

2.3.5
population mean
the average of all potential measurements in a given population weighted by their relative frequencies in the population

2.3.6
probability density function
function $f(x)$ is a probability density function for the continuous random variable X if

$$f(x) \geqslant 0 \tag{1}$$

and

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{2}$$

NOTE The probability that the random variable X assumes a value between a and b is given by

$$Pr(a < X < b) = \int_{a}^{b} f(x)\,dx \tag{3}$$

2.3.7
ranking estimator
function that estimates the probability of failure to a particular strength measurement within a ranked sample

2.3.8
sample
collection of measurements or observations taken from a specified population

2.3.9
skewness
term relating to the asymmetry of a probability density function

NOTE The distribution of failure strength for advanced ceramics is not symmetric with respect to the maximum value of the distribution function but has one tail longer than the other.

2.3.10
statistical bias
inherent to most estimates, this is a type of consistent numerical offset in an estimate relative to the true underlying value

NOTE The magnitude of the bias error typically decreases as the sample size increases.

2.3.11
unbiased estimator
estimator that has been corrected for statistical bias error

2.4 Weibull distributions

2.4.1
Weibull distribution
continuous random variable X has a two-parameter Weibull distribution if the probability density function is given by

$$f(x) = \left(\frac{m}{\beta}\right)\left(\frac{x}{\beta}\right)^{m-1} \exp\left[-\left(\frac{x}{\beta}\right)^{m}\right] \text{ when } x > 0 \tag{4}$$

or

$$f(x) = 0 \text{ when } x \leqslant 0 \tag{5}$$

and the cumulative distribution function is given by

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\beta}\right)^{m}\right] \text{ when } x > 0 \tag{6}$$

or

$$F(x) = 0 \text{ when } x \leqslant 0 \tag{7}$$

where

m is the Weibull modulus (or the shape parameter) (> 0);

β is the Weibull scale parameter (> 0)

NOTE 1 The random variable representing uniaxial tensile strength of an advanced ceramic will assume only positive values, and the distribution is asymmetrical about the mean. These characteristics rule out the use of the normal distribution (as well as others) and point to the use of the Weibull and similar skewed distributions. If the random variable representing uniaxial tensile strength of an advanced ceramic is characterized by Equations 4 to 7, then the probability that this advanced ceramic will fail under an applied uniaxial tensile stress σ is given by the cumulative distribution function

$$P_{\mathrm{f}} = 1 - \exp\left[-\left(\frac{\sigma}{\sigma_{\theta}}\right)^{m}\right] \text{ when } \sigma > 0 \tag{8}$$

$$P_{\mathrm{f}} = 0 \text{ when } \sigma \leqslant 0 \tag{9}$$

where

P_{f} is the probability of failure;

σ_{θ} is the Weibull characteristic strength.

NOTE 2 The Weibull characteristic strength is dependent on the uniaxial test specimen (tensile, flexural, or pressurized ring) and will change with specimen geometry. In addition, the Weibull characteristic strength has units of stress, and should be reported using units of MPa or GPa.

NOTE 3 An alternative expression for the probability of failure is given by

$$P_\mathrm{f} = 1 - \exp\left[-\int_V \left(\frac{\sigma}{\sigma_0} \right)^m dV \right] \text{ when } \sigma > 0 \tag{10}$$

$$P_\mathrm{f} = 0 \text{ when } \sigma \leqslant 0 \tag{11}$$

The integration in the exponential is performed over all tensile regions of the specimen volume if the strength-controlling flaws are randomly distributed through the volume of the material, or over all tensile regions of the specimen area if flaws are restricted to the specimen surface. The integration is sometimes carried out over an effective gauge section instead of over the total volume or area. In Equation 10, σ_0 is the Weibull material scale parameter and can be described as the Weibull characteristic strength of a specimen with unit volume or area loaded in uniform uniaxial tension. The Weibull material scale parameter has units of stress·(volume)$^{1/m}$, and should be reported using units of MPa·m$^{3/m}$ or GPa·m$^{3/m}$ if the strength-controlling flaws are distributed through the volume of the material. If the strength-controlling flaws are restricted to the surface of the specimens in a sample, then the Weibull material scale parameter should be reported using units of MPa·m$^{2/m}$ or GPa·m$^{2/m}$. For a given specimen geometry, Equations 8 and 10 can be combined, to yield an expression relating σ_0 and σ_θ. Further discussion related to this issue can be found in Annex A.

3 Symbols

A specimen area

b gauge section dimension, base of bend test specimen

d gauge section dimension, depth of bend test specimen

$f(x)$ probability density function

$F(x)$ cumulative distribution function

L likelihood function

L_i length of the inner load span for a bend test specimen

L_o length of the outer load span for a bend test specimen

m Weibull modulus

$\hat{m}$ estimate of the Weibull modulus

$\hat{m}_\mathrm{U}$ unbiased estimate of the Weibull modulus

N number of specimens in a sample

P_f probability of failure

r number of specimens that failed from the flaw population for which the Weibull estimators are being calculated

t intermediate quantity defined by Equation 15, used in calculation of confidence bounds

V specimen volume

x realization of a random variable X

X random variable

β Weibull scale parameter

σ uniaxial tensile stress

$\hat{\sigma}$ estimate of mean strength

σ_j maximum stress in the j th test specimen at failure

σ_0 Weibull material scale parameter (strength relative to unit size) defined in Equation 10

$\hat{\sigma}_0$ estimate of the Weibull material scale parameter

σ_θ Weibull characteristic strength (associated with a test specimen) defined in Equation 8

$\hat{\sigma}_\theta$ estimate of the Weibull characteristic strength

4 Significance and use

4.1 This International Standard enables the experimentalist to estimate Weibull distribution parameters from failure data. These parameters permit a description of the statistical nature of fracture of fine ceramic materials for a variety of purposes, particularly as a measure of reliability as it relates to strength data utilized for mechanical design purposes. The observed strength values are dependent on specimen size and geometry. Parameter estimates can be computed for a given specimen geometry $(\hat{m}, \hat{\sigma}_\theta)$, but it is suggested that the parameter estimates be transformed and reported as material-specific parameters $(\hat{m}, \hat{\sigma}_0)$. In addition, different flaw distributions (e.g., failures due to inclusions or machining damage) may be observed, and each will have its own strength distribution parameters. The procedure for transforming parameter estimates for typical specimen geometries and flaw distributions is outlined in Annex A.

4.2 This International Standard provides two approaches, Method A and Method B, which are appropriate for different purposes.

Method A provides a simple analysis for circumstances in which the nature of strength-defining flaws is either known or assumed to be from a single population. Fractography to identify and group test items with given flaw types is thus not required. This method is suitable for use for simple material screening.

Method B provides an analysis for the general case in which competing flaw populations exist. This method is appropriate for final component design and analysis. The method requires that fractography be undertaken to identify the nature of strength-limiting flaws and assign failure data to given flaw population types.

4.3 In method A, a strength data set can be analysed and values of the Weibull modulus and characteristic strength $(\hat{m}, \hat{\sigma}_\theta)$ are produced, together with confidence bounds on these parameters. If necessary the estimate of the mean strength can be computed. Finally, a graphical representation of the failure data along with a test report can be prepared. It should be noted that the confidence bounds are frequently widely spaced, which indicates that the results of the analysis should not be used to extrapolate far beyond the existing bounds of probability of failure.

4.4 In method B, begin by performing a fractographic examination of each failed specimen in order to characterize fracture origins. Screen the data associated with each flaw distribution for outliers. If all failures originate from a single flaw distribution compute an unbiased estimate of the Weibull modulus, and compute confidence bounds for both the estimated Weibull modulus and the estimated Weibull characteristic strength. If the failures originate from more than one flaw type, separate the data sets associated with each flaw type, and subject these individually to the censored analysis. Finally, prepare a graphical representation of the failure data along with a test report. When using the results of the analysis for design purposes it should be noted that there is an implicit assumption that the flaw populations in the strength test pieces and the components are of the same types.

5 Method A: maximum likelihood parameter estimators for single flaw populations

5.1 General

This International Standard outlines the application of parameter estimation methods based on the maximum likelihood technique. This technique has certain advantages. The parameter estimates obtained using the maximum likelihood technique are unique (for a two-parameter Weibull distribution), and as the size of the sample increases, the estimates statistically approach the true values of the population more efficiently than other parameter estimation techniques.

5.2 Censored data

The application of the techniques presented in this International Standard can be complicated by the presence of test specimens that fail from extraneous flaws, fractures that originate outside the effective gauge section, and unidentified fracture origins. If these complications arise, the strength data from these specimens should generally not be discarded. Strength data from specimens with fracture origins outside the effective gauge section [3] and from specimens with fractures that originate from extraneous flaws should be censored, and the maximum likelihood methods presented later in Method B (Clause 6) of this International Standard are applicable. It is imperative that the number of unidentified fracture origins, and how they were classified, be stated in the test report. A discussion of the appropriateness of each option can be found in 6.2.2.

5.3 Likelihood functions

The likelihood function for the two-parameter Weibull distribution of a sample with a single flaw population [4] is defined by the expression:

$$L = \prod_{i=1}^{N} \left(\frac{\hat{m}}{\hat{\sigma}_\theta} \right) \left(\frac{\sigma_i}{\hat{\sigma}_\theta} \right)^{\hat{m}-1} \exp\left[-\left(\frac{\sigma_i}{\hat{\sigma}_\theta} \right)^{\hat{m}} \right] \tag{12}$$

NOTE σ_i is the maximum stress in the i th test specimen at failure and N is the number of test specimens in the sample being analysed. The parameter estimates (the Weibull modulus, $\hat{m}$, and the characteristic strength, $\hat{\sigma}_\theta$) are determined by taking the partial derivatives of the logarithm of the likelihood function with respect to $\hat{m}$ and $\hat{\sigma}_\theta$ and equating the resulting expressions to zero.

The system of equations obtained by differentiating the log likelihood function for a sample with a single flaw population [5] is given by

$$\frac{\sum_{i=1}^{N} (\sigma_i)^{\hat{m}} \ln(\sigma_i)}{\sum_{i=1}^{N} (\sigma_i)^{\hat{m}}} - \frac{1}{N} \sum_{i=1}^{N} \ln(\sigma_i) - \frac{1}{\hat{m}} = 0 \tag{13}$$

and

$$\hat{\sigma}_\theta = \left[\left(\sum_{i=1}^{N} (\sigma_i)^{\hat{m}} \right) \frac{1}{N} \right]^{1/\hat{m}} \tag{14}$$

Equation 13 is solved first for $\hat{m}$. Subsequently $\hat{\sigma}_\theta$ is computed from Equation 14. Obtaining a closed form solution of Equation 13 for $\hat{m}$ is not possible. This expression must be solved numerically.

Since the characteristic strength also reflects specimen geometry and stress gradients, this International Standard suggests reporting the estimated Weibull material scale parameter, $\hat{\sigma}_0$. Expressions that relate $\hat{\sigma}_\theta$ to the Weibull material scale parameter σ_0 for typical specimen geometries are given in Annex A.

5.4 Bias correction

5.4.1 The procedures described herein, to correct for statistical bias errors and to compute confidence bounds, are appropriate only for data sets where all failures originate from a single population (i.e., an uncensored sample). Procedures for bias correction and confidence bounds in the presence of multiple active flaw populations are not currently well developed. The statistical bias associated with the estimator $\hat{\sigma}_\theta$ is minimal (< 0,3 % for 20 test specimens, as opposed to ≈ 7 % bias for $\hat{m}$ with the same number of specimens). Therefore, this International Standard allows the assumption that $\hat{\sigma}_\theta$ is an unbiased estimator of the true population parameter. The parameter estimate of the Weibull modulus, $\hat{m}$, generally exhibits statistical bias. The amount of statistical bias depends on the number of specimens in the sample. An unbiased estimate of $\hat{m}$ shall be obtained by multiplying $\hat{m}$ by unbiasing factors [6]. This procedure is discussed in 5.4.2. Statistical bias associated with the maximum likelihood estimators presented in this International Standard can be reduced by increasing the sample size.

5.4.2 An unbiased estimator produces nearly zero statistical bias between the value of the true parameter and the point estimate. The amount of deviation can be quantified either as a percent difference or with unbiasing factors. In keeping with the accepted practice in the open literature, this International Standard quantifies statistical bias through the use of unbiasing factors, denoted here as UF. Depending on the number of specimens in a given sample, the point estimate of the Weibull modulus, $\hat{m}$, may exhibit significant statistical bias. An unbiased estimate of the Weibull modulus (denoted as $\hat{m}_U$) is obtained by multiplying the biased estimate with an appropriate unbiasing factor. Unbiasing factors for $\hat{m}$ are listed in Table 1. An example in Annex B demonstrates the use of Table 1 in correcting a biased estimate of the Weibull modulus.