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Standard Practice for Reporting Uniaxial Strength Data and Estimating Weibull Distribution Parameters for Advanced Ceramics[1]

This standard is issued under the fixed designation C 1239; the number immediately following the designation indicates the year of original adoption or, in the case of revision, the year of last revision. A number in parentheses indicates the year of last reapproval. A superscript epsilon (ϵ) indicates an editorial change since the last revision or reapproval.

1. Scope

1.1 This practice covers the evaluation and subsequent reporting of uniaxial strength data and the estimation of probability distribution parameters for advanced ceramics that fail in a brittle fashion. The failure strength of advanced ceramics is treated as a continuous random variable. Typically, a number of test specimens with well-defined geometry are failed under well-defined isothermal forcing conditions. The force at which each test specimen fails is recorded. The resulting failure stresses are used to obtain parameter estimates associated with the underlying population distribution. This practice is restricted to the assumption that the distribution underlying the failure strengths is the two-parameter Weibull distribution with size scaling. Furthermore, this practice is restricted to test specimens (tensile, flexural, pressurized ring, etc.) that are primarily subjected to uniaxial stress states. Section 8 outlines methods to correct for bias errors in the estimated Weibull parameters and to calculate confidence bounds on those estimates from data sets where all failures originate from a single flaw population (that is, a single failure mode). In samples where failures originate from multiple independent flaw populations (for example, competing failure modes), the methods outlined in Section 8 for bias correction and confidence bounds are not applicable.

1.2 Measurements of the strength at failure are taken for one of two reasons: either for a comparison of the relative quality of two materials, or the prediction of the probability of failure (or, alternatively, the fracture strength) for a structure of interest. This practice will permit estimates of the distribution parameters that are needed for either. In addition, this practice encourages the integration of mechanical property data and fractographic analysis.

1.3 This practice includes the following:

1.4 The values stated in SI units are to be regarded as the standard per IEEE/ASTM SI 10.

2. Referenced Documents

2.1 *ASTM Standards:* [2]

C 1145 Terminology of Advanced Ceramics

C 1322 Practice for Fractography and Characterization of Fracture Origins in Advanced Ceramics

E 6 Terminology Relating to Methods of Mechanical Testing

E 178 Practice for Dealing With Outlying Observations

E 456 Terminology Relating to Quality and Statistics

IEEE/ASTM SI 10 American National Standard for Use of the International System of Units (SI): The Modern Metric System

3. Terminology

3.1 Proper use of the following terms and equations will alleviate misunderstanding in the presentation of data and in the calculation of strength distribution parameters.

3.1.1 *censored strength data*—strength measurements (that is, a sample) containing suspended observations such as that produced by multiple competing or concurrent flaw populations.

3.1.1.1 Consider a sample where fractography clearly established the existence of three concurrent flaw distributions (although this discussion is applicable to a sample with any number of concurrent flaw distributions). The three concurrent flaw distributions are referred to here as distributions *A, B,* and *C*. Based on fractographic analyses, each test specimen strength is assigned to a flaw distribution that initiated failure.

[1] This practice is under the jurisdiction of ASTM Committee C28 on Advanced Ceramics and is the direct responsibility of Subcommittee C28.02 on Reliability.

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[2] For referenced ASTM standards, visit the ASTM website, www.astm.org, or contact ASTM Customer Service at service@astm.org. For *Annual Book of ASTM Standards* volume information, refer to the standard's Document Summary page on the ASTM website.

In estimating parameters that characterize the strength distribution associated with flaw distribution *A*, all test specimens (and not just those that failed from Type A flaws) must be incorporated in the analysis to ensure efficiency and accuracy of the resulting parameter estimates. The strength of a test specimen that failed by a Type B (or Type C) flaw is treated as a *right censored* observation relative to the *A* flaw distribution. Failure due to a Type B (or Type C) flaw restricts, or censors, the information concerning Type A flaws in a test specimen by suspending the test before failure occurred by a Type A flaw (**1**).[3] The strength from the most severe Type A flaw in those test specimens that failed from Type B (or Type C) flaws is higher than (and thus to the *right* of) the observed strength. However, no information is provided regarding the magnitude of that difference. Censored data analysis techniques incorporated in this practice utilize this incomplete information to provide efficient and relatively unbiased estimates of the distribution parameters.

3.2 *Definitions:*

3.2.1 *competing failure modes*—distinguishably different types of fracture initiation events that result from concurrent (competing) flaw distributions.

3.2.2 *compound flaw distributions*—any form of multiple flaw distribution that is neither pure concurrent nor pure exclusive. A simple example is where every test specimen contains the flaw distribution *A*, while some fraction of the test specimens also contains a second independent flaw distribution *B*.

3.2.3 *concurrent flaw distributions*—type of multiple flaw distribution in a homogeneous material where every test specimen of that material contains representative flaws from each independent flaw population. Within a given test specimen, all flaw populations are then present concurrently and are competing with each other to cause failure. This term is synonymous with "competing flaw distributions."

3.2.4 *effective gage section*—that portion of the test specimen geometry that has been included within the limits of integration (volume, area, or edge length) of the Weibull distribution function. In tensile test specimens, the integration may be restricted to the uniformly stressed central gage section, or it may be extended to include transition and shank regions.

3.2.5 *estimator*—well-defined function that is dependent on the observations in a sample. The resulting value for a given sample may be an estimate of a distribution parameter (a point estimate) associated with the underlying population. The arithmetic average of a sample is, for example, an estimator of the distribution mean.

3.2.6 *exclusive flaw distributions*—type of multiple flaw distribution created by mixing and randomizing test specimens from two or more versions of a material where each version contains a different single flaw population. Thus, each test specimen contains flaws exclusively from a single distribution, but the total data set reflects more than one type of strength-controlling flaw. This term is synonymous with "mixtures of flaw distributions."

3.2.7 *extraneous flaws*—strength-controlling flaws observed in some fraction of test specimens that cannot be present in the component being designed. An example is machining flaws in ground bend test specimens that will not be present in as-sintered components of the same material.

3.2.8 *fractography*—analysis and characterization of patterns generated on the fracture surface of a test specimen. Fractography can be used to determine the nature and location of the critical fracture origin causing catastrophic failure in an advanced ceramic test specimen or component.

3.2.9 *fracture origin*—the source from which brittle fracture commences (Terminology C 1145).

3.2.10 *multiple flaw distributions*—strength controlling flaws observed by fractography where distinguishably different flaw types are identified as the failure initiation site within different test specimens of the data set and where the flaw types are known or expected to originate from independent causes.

3.2.10.1 *Discussion*—An example of multiple flaw distributions would be carbon inclusions and large voids which may both have been observed as strength controlling flaws within a data set and where there is no reason to believe that the frequency or distribution of carbon inclusions created during fabrication was in any way dependent on the frequency or distribution of voids (or vice-versa).

3.2.11 *population*—totality of potential observations about which inferences are made.

3.2.12 *population mean*—average of all potential measurements in a given population weighted by their relative frequencies in the population.

3.2.13 *probability density function*—function $f(x)$ is a probability density function for the continuous random variable X if:

$$f(x) \geq 0 \tag{1}$$

and

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{2}$$

The probability that the random variable X assumes a value between a and b is given by the following equation:

$$Pr(a < X < b) = \int_{a}^{b} f(x)\,dx \tag{3}$$

3.2.14 *sample*—collection of measurements or observations taken from a specified population.

3.2.15 *skewness*—term relating to the asymmetry of a probability density function. The distribution of failure strength for advanced ceramics is not symmetric with respect to the maximum value of the distribution function but has one tail longer than the other.

3.2.16 *statistical bias*—inherent to most estimates, this is a type of consistent numerical offset in an estimate relative to the true underlying value. The magnitude of the bias error typically decreases as the sample size increases.

[3] The boldface numbers in parentheses refer to the list of references at the end of this practice.

3.2.17 *unbiased estimator*—estimator that has been corrected for statistical bias error.

3.2.18 *Weibull distribution*—continuous random variable X has a two-parameter Weibull distribution if the probability density function is given by the following equations:

$$f(x) = \left(\frac{m}{\beta}\right)\left(\frac{x}{\beta}\right)^{m-1} exp\left[-\left(\frac{x}{\beta}\right)^{m}\right] \quad x>0 \tag{4}$$

$$f(x) = 0 \quad x \leq 0 \tag{5}$$

and the cumulative distribution function is given by the following equations:

$$F(x) = 1 - exp\left[-\left(\frac{x}{\beta}\right)^{m}\right] \quad x > 0 \tag{6}$$

or

$$F(x) = 0 \quad x \leq 0 \tag{7}$$

where

m = Weibull modulus (or the shape parameter) (>0), and
β = scale parameter (>0).

3.2.19 The random variable representing uniaxial tensile strength of an advanced ceramic will assume only positive values, and the distribution is asymmetrical about the mean. These characteristics rule out the use of the normal distribution (as well as others) and point to the use of the Weibull and similar skewed distributions. If the random variable representing uniaxial tensile strength of an advanced ceramic is characterized by Eq 4-7, then the probability that this advanced ceramic will fail under an applied uniaxial tensile stress σ is given by the cumulative distribution function as follows:

$$P_f = 1 - exp\left[-\left(\frac{\sigma}{\sigma_\theta}\right)^{m}\right] \sigma > 0 \tag{8}$$

$$P_f = 0 \quad \sigma \leq 0 \tag{9}$$

where:

P_f = probability of failure, and
σ_θ = Weibull characteristic strength.

Note that the Weibull characteristic strength is dependent on the uniaxial test specimen (tensile, flexural, or pressurized ring) and will change with test specimen size and geometry. In addition, the Weibull characteristic strength has units of stress and should be reported using units of megapascals or gigapascals.

3.2.20 An alternative expression for the probability of failure is given by the following equation:

$$P_f = 1 - exp\left[-\int_v\left(\frac{\sigma}{\sigma_0}\right)^{m} dV\right] \sigma > 0 \tag{10}$$

$$P_f = 0 \quad \sigma \leq 0 \tag{11}$$

The integration in the exponential is performed over all tensile regions of the test specimen volume if the strength-controlling flaws are randomly distributed through the volume of the material, or over all tensile regions of the test specimen area if flaws are restricted to the test specimen surface. The integration is sometimes carried out over an effective gage section instead of over the total volume or area. In Eq 10, σ_0 is the Weibull material scale parameter. The parameter is a material property if the two-parameter Weibull distribution properly describes the strength behavior of the material. In addition, the Weibull material scale parameter can be described as the Weibull characteristic strength of a test specimen with unit volume or area forced in uniform uniaxial tension. The Weibull material scale parameter has units of stress·(volume)$^{1/m}$and should be reported using units of MPa·(m)$^{3/m}$ or GPa·(m)$^{3/m}$ if the strength-controlling flaws are distributed through the volume of the material. If the strength-controlling flaws are restricted to the surface of the test specimens in a sample, then the Weibull material scale parameter should be reported using units of MPa·(m)$^{2/m}$ or GPa·(m)$^{2/m}$. For a given test specimen geometry, Eq 8 and Eq 10 can be equated, which yields an expression relating σ_0 and σ_θ. Further discussion related to this issue can be found in 7.6.

3.3 For definitions of other statistical terms, terms related to mechanical testing, and terms related to advanced ceramics used in this practice, refer to Terminologies E 456, C 1145, and E 6 or to appropriate textbooks on statistics **(2-5)** .

3.4 *Symbols:*

A = test specimen area (or area of effective gage section, if used).
b = gage section dimension, base of bend test specimen.
d = gage section dimension, depth of bend test specimen.
$F(x)$ = cumulative distribution function.
$f(x)$ = probability density function.
L_i = length of the inner span for a bend test specimen.
L_o = length of the outer span for a bend test specimen.
$\mathscr{L}$ = likelihood function.
m = Weibull modulus.
$\hat{m}$ = estimate of the Weibull modulus.
$\hat{m}_U$ = unbiased estimate of the Weibull modulus.
N = number of test specimens in a sample.
P_f = probability of failure.
r = number of test specimens that failed from the flaw population for which the Weibull estimators are being calculated.
t = intermediate quantity defined by Eq 27, used in calculation of confidence bounds.
V = test specimen volume (or volume of effective gage section, if used).
X = random variable.
x = realization of a random variable X.
β = Weibull scale parameter.
ϵ = stopping tolerance in the computer algorithm MAXL.
$\hat{\mu}$ = estimate of mean strength.
σ = uniaxial tensile stress.
σ_i = maximum stress in the ith test specimen at failure.
σ_j = maximum stress in the jth test specimen at failure.
σ_O = Weibull material scale parameter (strength relative to unit size) defined in Eq 10.
σ_θ = Weibull characteristic strength (associated with a test specimen) defined in Eq 8.
$\hat{\sigma}_O$ = estimate of the Weibull material scale parameter.

$\hat{\sigma}_\theta$ = estimate of the Weibull characteristic strength.

4. Summary of Practice

4.1 This practice enables the experimentalist to estimate Weibull distribution parameters from failure data. Begin by performing a fractographic examination of each failed test specimen (optional, but highly recommended) in order to characterize fracture origins. Usually discrete fracture origins can be grouped by flaw distributions. Screen the data associated with each flaw distribution for outliers. Compute estimates of the biased Weibull modulus and Weibull characteristic strength. If necessary, compute the estimate of the mean strength. If all failures originate from a single flaw distribution, compute an unbiased estimate of the Weibull modulus and compute confidence bounds for both the estimated Weibull modulus and the estimated Weibull characteristic strength. Prepare a graphical representation of the failure data along with a test report.

5. Significance and Use

5.1 Advanced ceramics usually display a linear stress-strain behavior to failure. Lack of ductility combined with flaws that have various sizes and orientations leads to scatter in failure strength. Strength is not a deterministic property but instead reflects an intrinsic fracture toughness and a distribution (size and orientation) of flaws present in the material. This practice is applicable to brittle monolithic ceramics that fail as a result of catastrophic propagation of flaws present in the material. This practice is also applicable to composite ceramics that do not exhibit any appreciable bilinear or nonlinear deformation behavior. In addition, the composite must contain a sufficient quantity of uniformly distributed reinforcements such that the material is effectively homogeneous. Whisker-toughened ceramic composites may be representative of this type of material.

5.2 Two- and three-parameter formulations exist for the Weibull distribution. This practice is restricted to the two-parameter formulation. An objective of this practice is to obtain point estimates of the unknown parameters by using well-defined functions that incorporate the failure data. These functions are referred to as estimators. It is desirable that an estimator be consistent and efficient. In addition, the estimator should produce unique, unbiased estimates of the distribution parameters (**6**). Different types of estimators exist, including moment estimators, least-squares estimators, and maximum likelihood estimators. This practice details the use of maximum likelihood estimators due to the efficiency and the ease of application when censored failure populations are encountered.

5.3 Tensile and flexural test specimens are the most commonly used test configurations for advanced ceramics. The observed strength values are dependent on test specimen size and geometry. Parameter estimates can be computed for a given test specimen geometry ($\hat{m}$, $\hat{\sigma}_\theta$), but it is suggested that the parameter estimates be transformed and reported as material-specific parameters ($\hat{m}$, $\hat{\sigma}_0$). In addition, different flaw distributions (for example, failures due to inclusions or machining damage) may be observed, and each will have its own strength distribution parameters. The procedure for transforming parameter estimates for typical test specimen geometries and flaw distributions is outlined in 7.6.

5.4 Many factors affect the estimates of the distribution parameters. The total number of test specimens plays a significant role. Initially, the uncertainty associated with parameter estimates decreases significantly as the number of test specimens increases. However, a point of diminishing returns is reached when the cost of performing additional strength tests may not be justified. This suggests that a practical number of strength tests should be performed to obtain a desired level of confidence associated with a parameter estimate. The number of test specimens needed depends on the precision required in the resulting parameter estimate. Details relating to the computation of confidence bounds (directly related to the precision of the estimate) are presented in 8.3 and 8.4.

6. Outlying Observations

6.1 Before computing the parameter estimates, the data should be screened for outlying observations (outliers). An outlying observation is one that deviates significantly from other observations in the sample. It should be understood that an apparent outlying observation may be an extreme manifestation of the variability of the strength of an advanced ceramic. If this is the case, the data point should be retained and treated as any other observation in the failure sample. However, the outlying observation may be the result of a gross deviation from prescribed experimental procedure or an error in calculating or recording the numerical value of the data point in question. When the experimentalist is clearly aware that a gross deviation from the prescribed experimental procedure has occurred, the outlying observation may be discarded, unless the observation can be corrected in a rational manner. The procedures for dealing with outlying observations are detailed in Practice E 178.

7. Maximum Likelihood Parameter Estimators for Competing Flaw Distributions

7.1 This practice outlines the application of parameter estimation methods based on the maximum likelihood technique. This technique has certain advantages, especially when parameters must be determined from censored failure populations. When a sample of test specimens yields two or more distinct flaw distributions, the sample is said to contain censored data, and the associated methods for censored data must be employed. Fractography (see Section 9) should be used to determine whether multiple flaw distributions are present. The methods described in this practice include censoring techniques that apply to multiple concurrent flaw distributions. However, the techniques for parameter estimation presented in this practice are not directly applicable to data sets that contain exclusive or compound multiple flaw distributions (**7**). The parameter estimates obtained using the maximum likelihood technique are unique (for a two-parameter Weibull distribution), and as the size of the sample increases, the estimates statistically approach the true values of the population.

7.2 This practice allows failure to be controlled by multiple flaw distributions. Advanced ceramics typically contain two or more active flaw distributions each with an independent set of parameter estimates. The censoring techniques presented herein require positive confirmation of multiple flaw distributions, which necessitates fractographic examination to characterize the fracture origin in each test specimen. Multiple flaw distributions may be further evidenced by deviation from the linearity of the data from a single Weibull distribution (for example, Fig. 1). However, since there are many exceptions, observations of approximately linear behavior should not be considered sufficient reason to conclude that only a single flaw distribution is active.

7.2.1 For data sets with multiple active flaw distributions where one flaw distribution (identified by fractographic analysis) occurs in a small number of test specimens, it is sufficient to report the existence of this flaw distribution (and the number of occurrences), but it is not necessary to estimate Weibull parameters. Estimates of the Weibull parameters for this flaw distribution would be potentially biased with wide confidence bounds (neither of which could be quantified through use of this practice). However, special note should be made in the report if the occurrences of this flaw distribution take place in the upper or lower tail of the sample strength distribution.

7.3 The application of the censoring techniques presented in this practice can be complicated by the presence of test specimens that fail from extraneous flaws, fractures that originate outside the effective gage section, and unidentified fracture origins. If these complications arise, the strength data from these test specimens should generally not be discarded. Strength data from test specimens with fracture origins outside the effective gage section (**8**), and test specimens with fractures that originate from extraneous flaws should be censored; and the maximum likelihood methods presented in this practice are applicable.

NOTE 1—In this standard the gage section in four-point flexure is taken to mean the region between the two outer forceing rollers.

7.3.1 Test specimens with unidentified fracture origins sometimes occur. It is imperative that the number of unidentified fracture origins, and how they were classified, be stated in the test report. This practice recognizes four options the experimentalist can pursue when unidentified fracture origins are encountered during fractographic examinations. The situation may arise where more than one option will be used within a single data set. Test specimens with unidentified fracture origins can be:

7.3.1.1 *Option a*—Assigned a previously identified flaw distribution using inferences based on all available fractographic information,

7.3.1.2 *Option b*—Assigned the same flaw distribution as that of the test specimen closest in strength,

7.3.1.3 *Option c*—Assigned a new and as yet unspecified flaw distribution, and

7.3.1.4 *Option d*—Be removed from the sample.

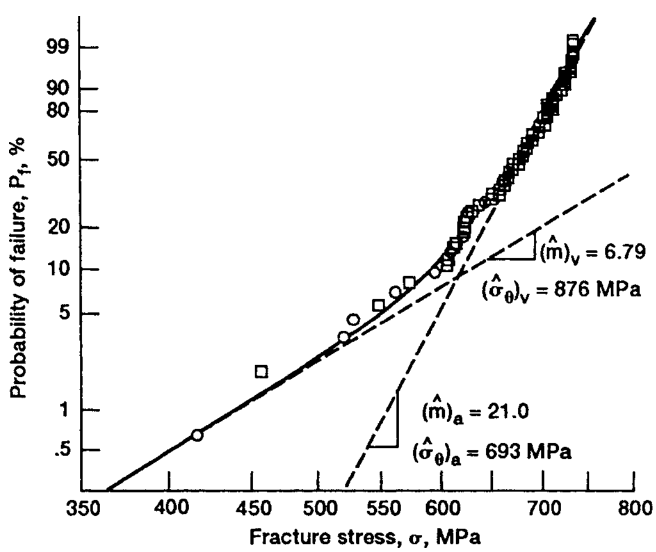
NOTE 2—The user is cautioned that the use of any of the options outlined in 7.3.1 for the classification of test specimens with unidentified fracture origins may create a consistent bias error in the parameter estimates. In addition, the magnitude of the bias cannot be determined by the methods presented in 8.2

7.3.2 A discussion of the appropriateness of each option in 7.3.1 is given in Appendix X2. If the strength data and the resulting parameter estimates are used for component design, the engineer must consult with the fractographer before and after performing the fractographic examination. Considerable judgement may be needed to identify the correct option. Whenever partial fractographic information is available, 7.3.1.1 is strongly recommended, especially if the data are used for component design. Conversely, 7.3.1.4 is not recommended by this practice unless there is overwhelming justification.

7.4 The likelihood function for the two-parameter Weibull distribution of a censored sample is defined by the following equation (**9**):

$$\mathcal{L}\left\{\prod_{i=1}^{r}\left(\frac{\hat{m}}{\hat{\sigma}_\theta}\right)\left(\frac{\sigma_i}{\hat{\sigma}_\theta}\right)^{\hat{m}-1} exp\left[-\left(\frac{\sigma_i}{\hat{\sigma}_\theta}\right)^{\hat{m}}\right]\right\}\prod_{j=r+1}^{N} exp\left[-\left(\frac{\sigma_j}{\hat{\sigma}_\theta}\right)^{\hat{m}}\right] \quad (12)$$

This expression is applied to a sample where two or more active concurrent flaw distributions have been identified from fractographic inspection. For the purpose of the discussion here, the different distributions will be identified as flaw Types A, B, C, etc. When Eq 12 is used to estimate the parameters associated with the A flaw distribution, then *r* is the number of test specimens where Type A flaws were found at the fracture origin, and *i* is the associated index in the first summation. The second summation is carried out for all other test specimens not failing from type A flaws (that is, Type B flaws, Type C flaws, etc.). Therefore, the sum is carried out from ($j = r + 1$) to N (the total number of test specimens) where j is the index in the second summation. Accordingly, σ_i and σ_j are the maximum stress in the *i*th and *j*th test specimen at failure. The parameter estimates (the Weibull modulus $\hat{m}$ and the characteristic strength $\hat{\sigma}_\theta$) are determined by taking the partial derivatives of the logarithm of the likelihood function with respect to $\hat{m}$ and $\hat{\sigma}_\theta$ and equating the resulting expressions to zero. Note that $\hat{\sigma}_\theta$ is a function of test specimen geometry and the estimate of the Weibull modulus. Expressions that relate $\hat{\sigma}_\theta$ to the Weibull material scale parameter $\hat{\sigma}_0$ for typical test specimen geometries are given in 7.6. Finally, the likelihood function for the



NOTE 1—The boxes refer to surface flaws; the circles refer to volume flaws.

FIG. 1 Example—Failure Data in Section 10.2

two-parameter Weibull distribution for a single-flaw population is defined by the following equation:

$$\mathcal{L} = \prod_{i=1}^{N} \left(\frac{\hat{m}}{\hat{\sigma}_\theta}\right)\left(\frac{\sigma_i}{\hat{\sigma}_\theta}\right)^{\hat{m}-1} exp\left[-\left(\frac{\sigma_i}{\hat{\sigma}_\theta}\right)^{\hat{m}}\right] \quad (13)$$

where r was taken equal to N in Eq 12.

7.5 The system of equations obtained by maximizing the log likelihood function for a censored sample is given by the following equations (**10**):

$$\frac{\sum_{i=1}^{N}(\sigma_i)^{\hat{m}}\ 1n(\sigma_i)}{\sum_{i=1}^{N}(\sigma_i)^{\hat{m}}} - \frac{1}{r}\sum_{i=1}^{r} 1n(\sigma_i) - \frac{1}{\hat{m}} = 0 \quad (14)$$

and

$$\hat{\sigma}_\theta = \left[\left(\sum_{i=1}^{N}(\sigma_i)^{\hat{m}}\right)\frac{1}{r}\right]^{1/\hat{m}} \quad (15)$$

where:

r = number of failed test specimens from a particular group of a censored sample.

When a sample does not require censoring, r is replaced by N in Eq 14 and Eq 15. Eq 14 is solved first for $\hat{m}$. Subsequently $\hat{\sigma}_\theta$ is computed from Eq 15. Obtaining a closed-form solution of Eq 14 for $\hat{m}$ is not possible. This expression must be solved numerically. When there are multiple active flaw populations, Eq 14 and Eq 15 must be solved for each flaw population. A computer algorithm (entitled MAXL) that calculates the root of Eq 14 is presented as a convenience in Appendix X1.

7.6 The numerical procedure in accordance with 7.5 yields parameter estimates of the Weibull modulus ($\hat{m}$) and the characteristic strength ($\hat{\sigma}_\theta$). Since the characteristic strength also reflects test specimen geometry and stress gradients, this standard suggests reporting the estimated Weibull material scale parameter $\hat{\sigma}_0$.

7.6.1 The following equation defines the relationship between the parameters for tensile test specimens:

$$(\hat{\sigma}_0)_V = (V)^{1/(\hat{m})_V}(\hat{\sigma}_\theta)_V \quad (16)$$

where V is the volume of the uniform gage section of the tensile test specimen, and the fracture origins are spatially distributed strictly within this volume. The gage section of a tensile test specimen is defined herein as the central region of the test specimen with the smallest constant cross-sectional area. However, the experimentalist may include transition regions and the shank regions of the test specimen if the volume (or area) integration defined by Eq 10 is analyzed properly. This procedure is discussed in 7.6.3. If the transition region or the shank region, or both, are included in the integration, Eq 16 is not applicable. For tensile test specimens in which fracture origins are spatially distributed strictly at the surface of the test specimens tested, the following equation applies:

$$(\hat{\sigma}_0)_A = (A)^{1/(\hat{m})_A}(\hat{\sigma}_\theta)_A \quad (17)$$

where A = surface area of the uniform gage section.

7.6.2 For flexural test specimen geometries, the relationships become more complex (**11**). The following relationship is based on the geometry of a flexural test specimen found in Fig. 2. For fracture origins spatially distributed strictly within both the volume of a flexural test specimen and the outer span, the following equation applies:

$$(\hat{\sigma}_0)_V = (\hat{\sigma}_\theta)_V \left|\frac{V\left[\left(\frac{L_i}{L_o}\right)(\hat{m})_v + 1\right]}{2[(\hat{m})_v + 1]^2}\right|^{1/(\hat{m})V} \quad (18)$$

where:

L_i = length of the inner span,
L_o = length of the outer span,
V = volume of the gage section defined by the following expression:

$$V = b\,d\,L_o \quad (19)$$

and:

b,d = dimensions identified in Fig. 2.

For fracture origins spatially distributed strictly at the surface of a flexural test specimen and within the outer span, the following equation applies:

$$(\hat{\sigma}_0)_A = (\hat{\sigma}_\theta)_A\left[L_o\left(\frac{d}{(\hat{m})_A + 1} + b\right)\left(\frac{\left(\frac{L_i}{L_o}\right)(\hat{m})_A + 1}{(\hat{m})_A + 1}\right)\right]^{1/(\hat{m})A} \quad (20)$$

7.6.3 Test specimens other than tensile and flexure test specimens may be utilized. Relationships between the estimate of the Weibull characteristic strength and the Weibull material scale parameter for any test specimen configuration can be derived by equating the expressions defined by Eq 8 and Eq 10 with the modifications that follow. Begin by replacing σ (an applied uniaxial tensile stress) in Eq 8 with σ_{max}, which is defined as the maximum tensile stress within the test specimen of interest. Thus:

$$P_f = 1 - exp\left[-\left(\frac{\sigma_{max}}{\sigma_\theta}\right)^m\right] \quad (21)$$

Also perform the integration given in Eq 10 such that

$$P_f = 1 - exp\left[-kV\left(\frac{\sigma_{max}}{\sigma_0}\right)^m\right] \quad (22)$$

where k is a dimensionless constant that accounts for test specimen geometry and stress gradients. Note that in general, k is a function of the estimated Weibull modulus m, and is always less than or equal to unity. The product (kV) is often referred to as the effective volume (with the designation V_E). The effective volume can be interpreted as the size of an

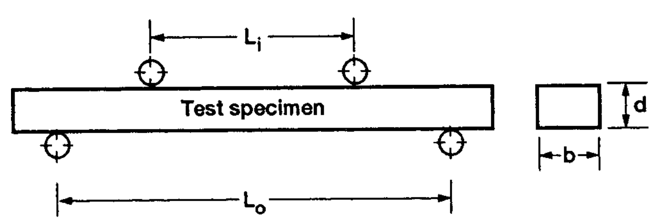


FIG. 2 Flexural Test Specimen Geometry