5 Abstract coordinate systems

5.1 Introduction

An abstract coordinate system is a means of identifying positions in position-space by coordinate n-tuples. An abstract coordinate system is completely defined in terms of the mathematical structure of position-space. In this International Standard the term “coordinate system”, if not otherwise qualified, is defined to mean “abstract coordinate system.” Each coordinate system has a coordinate system type (see 5.4). Other coordinate system related concepts defined in this clause include coordinate-component surfaces and coordinate-component curves (see 5.5), linearity and other properties (see 5.6), and localization (see 5.7). Map projections and augmented map projections are defined and treated as special cases of the general abstract coordinate system concept (see 5.8). Standardized abstract coordinate systems are specified in 5.9.

In Clause 6 a temporal coordinate system is defined as a means of identifying events in the time continuum by coordinate 1-tuples using an abstract coordinate system of coordinate system type 1D. In Clause 8 a spatial coordinate system is defined as an abstract coordinate system suitably combined with a normal embedding (see Clause 7) as a means of identifying points in object-space by coordinate n-tuples.

5.2 Preliminaries

This International Standard takes a functional approach to the construction of coordinate systems. Annex A provides a concise summary of mathematical concepts and specifies the notational conventions used in this International Standard. In particular, Annex A defines the terms interior, one-to-one, smooth, smooth surface, smooth curve, orientation-preserving, and connected. The concept of $\mathbf{R}^n$ as a vector space, the point-set topology of $\mathbf{R}^n$, and the theory of real-valued functions on $\mathbf{R}^n$ are all assumed. Algebraic and analytic geometry, including the concepts of point, line, and plane, are also assumed. Together with such common concepts, a newly introduced concept replete will be used. A set D is replete if all points in D belong to the closure of the interior of D (see Annex A). A replete set is a generalization of an open set that allows the inclusion of boundary points. Boundary points are important in the definitions of certain coordinate systems.

5.3 Abstract CS

An *abstract Coordinate System* (CS) is a means of identifying a set of positions in an abstract Euclidean space that shall be comprised of:

a) a CS domain,

b) a generating function, and

c) a CS range,

where:

a) The *CS domain* shall be a connected replete domain in the Euclidean space of n-tuples ($1 \le n \le m$), called the *coordinate-space.*

b) The *generating function* shall be a one-to-one, smooth, orientation-preserving function from the CS domain onto the CS range.

c) The *CS range* shall be a set of positions in a Euclidean space of dimension m ($n \le m \le 3$), called the *position-space.* When $n = 2$ and $m = 3$, the CS range shall be a subset of a smooth surface[5]. When $n = 1$ and $m = 2$ or 3, the CS range shall be a subset of an implicitly specified smooth curve[6].

An element of the CS domain shall be called a *coordinate*[7]. The k^{th}-component of a coordinate n-tuple ($1 \le k \le n$) may be called the k^{th} *coordinate-component*. *Coordinate-component*[8] is the collective term for any k^{th} coordinate-component.

An element of the CS range shall be called a *position.* The *coordinate of a position* $\boldsymbol{p}$ shall be the unique coordinate whose generating function value is $\boldsymbol{p}$.

The generating function may be parameterized. The generating function parameters (if any) shall be called the *CS parameters.*

The inverse of the generating function shall be called the *inverse generating function*. The inverse generating function is one-to-one and is smooth and orientation-preserving in the interior of its domain, except at points in the image of the CS domain boundary points where it may be discontinuous. A CS may equivalently be defined by specifying the inverse generating function when the CS domain is an open set.

NOTE 1 The generating function of a CS is often specified by an algebraic and/or trigonometric description of a geometric relationship (see 5.3 Example). There are also CSs that do not have geometric derivations. The Mercator map projection (see Table 5.18) is specified to satisfy a functional requirement of conformality (see 5.8.3.2) rather than by a geometric construction.

EXAMPLE Polar CS: Considering the polar geometry depicted in Figure 5.1, define a generating function $\boldsymbol{F}$ as:

$$\boldsymbol{F}(\rho,\ \theta) = (x,\ y)$$

where:

$$x = \rho\cos(\theta),\ \text{and}\ \ y = \rho\sin(\theta).$$

The CS domain of $\boldsymbol{F}$ in coordinate-space is $\{(\rho,\theta)\ \text{in}\ \mathbf{R}^2 \mid 0 < \rho,\ 0 \le \theta < 2\pi\} \cup \{(0,0)\}$.
The CS range of $\boldsymbol{F}$ in position-space is $\mathbf{R}^2$.

This generating function is illustrated in Figure 5.2. The grey boxes with lighter grey edges in this figure represent the fact that the range in position-space extends indefinitely, and that the domain in coordinate-space extends indefinitely along the ρ-axis. The dotted grey edges indicate an open boundary. This CS range, CS domain, and generating function define an abstract CS representing polar coordinates as defined in mathematics [EDM, "Coordinates"]. The normative definition of the polar CS may be found in Table 5.33.

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[5] The generating function properties and the implicit function theorem together imply that for each point in the interior of the CS domain, there is an open neighbourhood of the point whose image under the generating function lies in a smooth surface. This requirement specifies that there exists one smooth surface for all of the points in the CS domain. This requirement is specified to exclude mathematically pathological cases.

[6] The generating function properties and the implicit function theorem together imply that for each point in the interior of the CS domain, there is an open neighbourhood of the point whose image under the generating function lies in a smooth curve. This requirement specifies that there exists one implicitly-defined smooth curve for all the points in the CS domain. This requirement is specified to exclude mathematically pathological cases.

[7] The ISO 19111 term for this concept is "coordinate tuple".

[8] The ISO 19111 term for this concept is "coordinate".

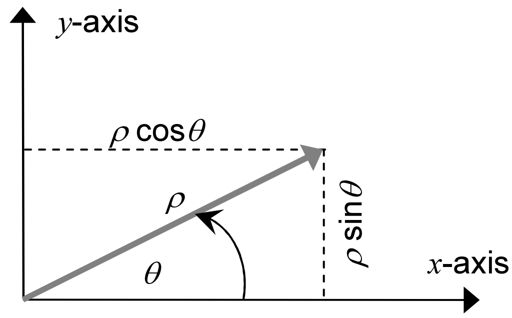

Figure 5.1 — Polar CS geometry

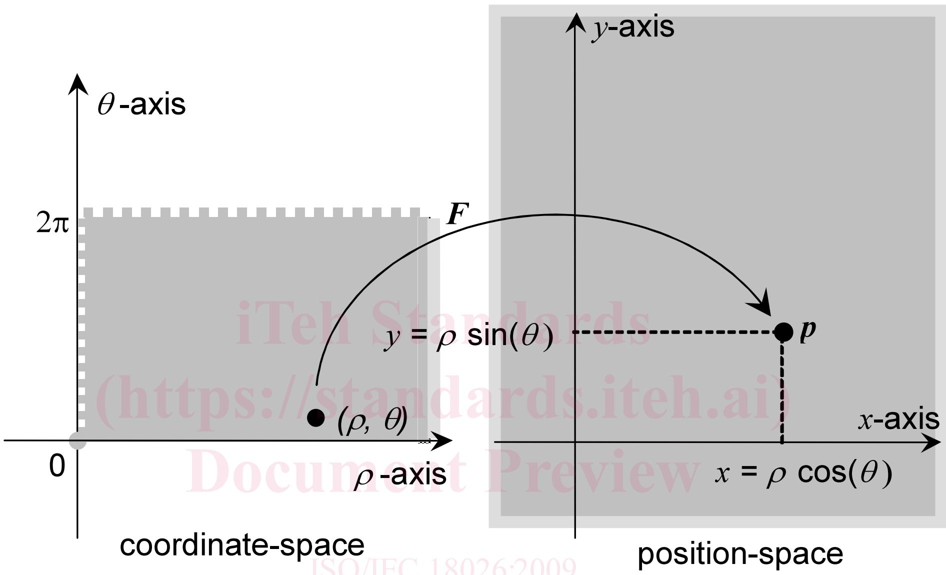

Figure 5.2 — The polar CS generating function

NOTE 2 In the special case where 1) the CS domain and CS range are both $\mathbf{R}^n$ and 2) the function is the identity function, this approach to defining coordinate systems reduces to the usual definition of the Euclidean coordinate system on $\mathbf{R}^n$ where each point is identified by an n-tuple of real numbers [EDM] (see Table 5.8, Table 5.29 and Table 5.35).

NOTE 3 The CS generating function has an inverse because it is one-to-one, but the inverse may be discontinuous at points in the image of CS domain boundary points. This is the case for the positive x-axis in the example above.

5.4 CS types

The coordinate-space and position-space dimensions characterize an abstract CS by *CS type* as defined in Table 5.1.

Table 5.1 — CS types

CS type	Dimension of coordinate-space	Dimension of position-space
3D	3	3

CS type	Dimension of coordinate-space	Dimension of position-space
surface	2	3
curve[9]	1	3
2D	2	2
plane curve[9]	1	2
1D	1	1

A CS of CS type 3D may be called a 3D CS, a CS of CS type surface may be called a surface CS, and a CS of CS type 2D may be called a 2D CS.

5.5 Coordinate surfaces, induced surface CSs, and coordinate curves

5.5.1 Introduction

The generating function of a 3D CS is a function of the three coordinate-components of a coordinate 3-tuple. If one of the coordinate-components is held fixed (to a constant value), then the generating function thus restricted to two variables may be viewed as a surface CS generating function (with a surface CS range). If two of the three coordinate-components are held fixed, the generating function restricted to one variable may be viewed as a curve CS generating function (with curve CS range). These observations motivate the definitions of coordinate-component surfaces and curves. The coordinate-component surface and coordinate-component curve concepts are required to specify induced CS relationships, for the definition of special coordinate curves parallel and meridian, and the definition of CS handedness (see also 10.5).

5.5.2 Coordinate-component surfaces and induced surface CSs

If $\boldsymbol{F}$ is the generating function of a 3D CS, and $\boldsymbol{u} = (u_0, v_0, w_0)$ is in the interior of the CS domain D, then three surface CS generating functions at $\boldsymbol{u}$ are defined by:

$\boldsymbol{S}_1(v,w) = \boldsymbol{F}(u_0,v,w)$,

$\boldsymbol{S}_2(u,w) = \boldsymbol{F}(u,v_0,w)$, and

$\boldsymbol{S}_3(u,v) = \boldsymbol{F}(u,v,w_0)$.

The CS domain for $\boldsymbol{S}_1$ is the connected component of $\{(v,w) \text{ in } \mathbf{R}^2 \mid (u_0,v,w) \text{ in } D\}$ which contains (v_0, w_0).

The CS domain for $\boldsymbol{S}_2$ is the connected component of $\{(u,w) \text{ in } \mathbf{R}^2 \mid (u,v_0,w) \text{ in } D\}$ which contains (u_0, w_0).

The CS domain for $\boldsymbol{S}_3$ is the connected component of $\{(u,v) \text{ in } \mathbf{R}^2 \mid (u,v,w_0) \text{ in } D\}$ which contains (u_0, v_0).

Each of these surface CSs shall be called, respectively, the 1st, 2nd, and 3rd *surface CS induced by* $\boldsymbol{F}$ at $\boldsymbol{u}$.

The CS ranges of these surface CSs are, respectively, the 1st, 2nd, and 3rd *coordinate-component surface at* $\boldsymbol{u}$.

EXAMPLE 1 Coordinate-component surface: The geodetic 3D CS with generating function $\boldsymbol{F}(\lambda,\varphi,h) = (x,y,z)$ is specified in Table 5.14 with CS parameters a and b. The 3rd coordinate-component surface at $\boldsymbol{u} = (\lambda_0,\varphi_0,0)$ is the surface of the oblate ellipsoid with major semi-axis a and minor semi-axis b.

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[9] The ISO 19111 concept of a linear reference system is a specialization of the curve CS and plane curve CS concepts.

EXAMPLE 2 Induced surface CS: The surface geodetic CS is specified in Table 5.24. Its CS domain, CS range and generating function are identical to the 3rd surface CS induced by the geodetic 3D generating function at $\boldsymbol{u} = (0,0,0)$. If h is replaced by 0 in the formulae for the generating and inverse generating functions of the geodetic 3D CS, they reduce to the surface geodetic formulae.

5.5.3 Coordinate-component curves

Coordinate-component curves are defined for CSs of CS type 3D, CS type surface, and CS type 2D.

The CS type 3D case:

If $\boldsymbol{F}$ is the generating function of a CS of CS type 3D, D is the CS domain, and $\boldsymbol{u} = (u_0, v_0, w_0)$ is in the interior of D, then the 1st, 2nd, and 3rd *coordinate-component curves at* $\boldsymbol{u}$ are parametrically specified, respectively, by the following smooth functions:

$$\boldsymbol{C}_1(u) = \boldsymbol{F}(u, v_0, w_0),$$

$$\boldsymbol{C}_2(v) = \boldsymbol{F}(u_0, v, w_0), \text{ and}$$

$$\boldsymbol{C}_3(w) = \boldsymbol{F}(u_0, v_0, w).$$

The domain for $\boldsymbol{C}_1$ is the connected component of $\{u \text{ in } \mathbf{R} \mid (u, v_0, w_0) \text{ in } D\}$ which contains u_0.

The domain for $\boldsymbol{C}_2$ is the connected component of $\{v \text{ in } \mathbf{R} \mid (u_0, v, w_0) \text{ in } D\}$ which contains v_0.

The domain for $\boldsymbol{C}_3$ is the connected component of $\{w \text{ in } \mathbf{R} \mid (u_0, v_0, w) \text{ in } D\}$ which contains w_0.

NOTE The intersection of two coordinate surfaces at u is (the locus of) a coordinate-component curve: $C_1 = S_2 \cap S_3$, $C_2 = S_1 \cap S_3$, $C_3 = S_1 \cap S_2$.

The CS type surface and CS type 2D cases:

If $\boldsymbol{F}$ is the generating function of a CS of CS type surface or CS type 2D, D is the CS domain, and $\boldsymbol{u} = (u_0, v_0)$ is in the interior of D, then the 1st and 2nd *coordinate-component curves at* $\boldsymbol{u}$ are parametrically specified, respectively, by the following smooth functions:

$$\boldsymbol{C}_1(u) = \boldsymbol{F}(u, v_0), \text{ and}$$

$$\boldsymbol{C}_2(v) = \boldsymbol{F}(u_0, v).$$

The domain for $\boldsymbol{C}_1$ is the connected component of $\{u \text{ in } \mathbf{R} \mid (u, v_0) \text{ in } D\}$ which contains u_0.

The domain for $\boldsymbol{C}_2$ is the connected component of $\{v \text{ in } \mathbf{R} \mid (u_0, v) \text{ in } D\}$ which contains v_0.

EXAMPLE If $\boldsymbol{u} = (\rho_0, \theta_0)$ is in the interior of the CS domain of the polar CS generating function $\boldsymbol{F}$ of the 5.3 Example, then the first coordinate-component curve is $\boldsymbol{C}_1(\theta) = \boldsymbol{F}(\rho_0, \theta) = (\rho_0 \cos\theta, \rho_0 \sin\theta)$, and the 2nd coordinate-component curve is $\boldsymbol{C}_2(\theta) = \boldsymbol{F}(\rho, \theta_0) = (\rho \cos\theta_0, \rho \sin\theta_0)$.

If $\boldsymbol{F}$ is the generating function for the geodetic 3D CS or the surface geodetic CS, and $\boldsymbol{u} = (\lambda_0, \varphi_0, 0)$ in the 3D case or $\boldsymbol{u} = (\lambda_0, \varphi_0)$ in the surface case, then (see Figure 5.3):

a) the 1st coordinate-component curve at $\boldsymbol{u}$ shall be called the *parallel* at $\boldsymbol{u}$, and

b) the 2nd coordinate-component curve at $\boldsymbol{u}$ shall be called the *meridian*[10] at $\boldsymbol{u}$.

The meridian at $\boldsymbol{u} = (0,0,0)$ or $(0,0)$ shall be called the *prime meridian*[11].

The parallel at $\boldsymbol{u} = (0,0,0)$ or $(0,0)$ shall be called the *equator*.

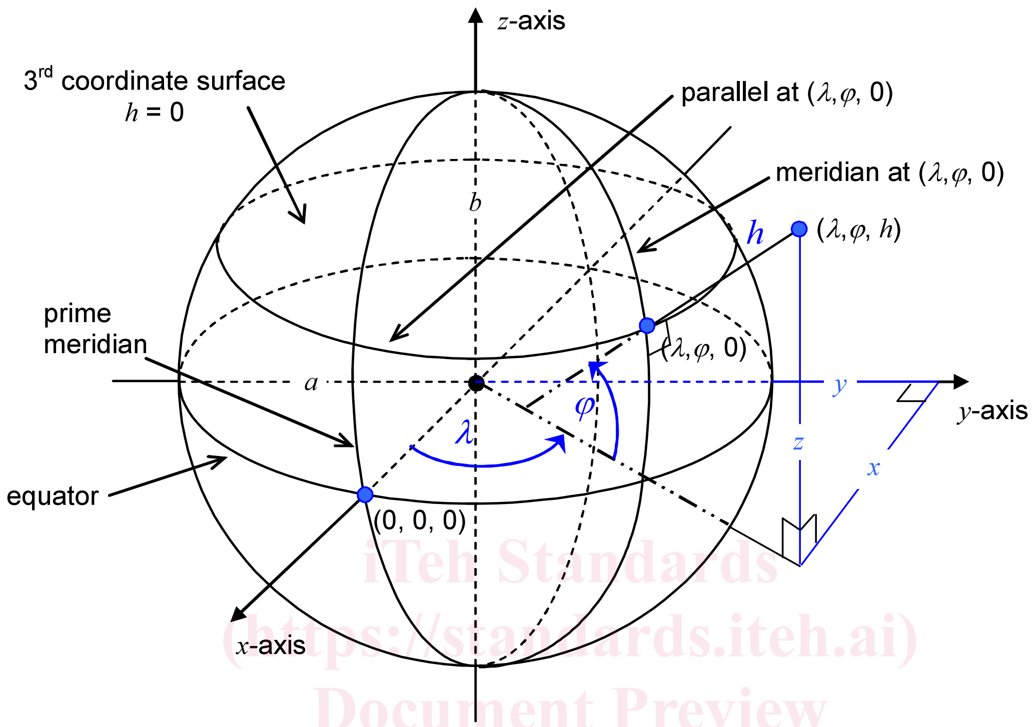

Figure 5.3 — Geodetic 3D CS geometry, and coordinate-component surface and curves

5.6 CS properties

5.6.1 Linearity

A CS with generating function $\boldsymbol{F}$ is a *linear CS* if $\boldsymbol{F}$ is an affine function. The CS domain of a linear coordinate system is all of the coordinate-space $\mathbf{R}^n$.

A *curvilinear CS* is a non-linear CS.

EXAMPLE The polar CS of 5.3 EXAMPLE is a curvilinear CS of CS type 2D.

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[10] ISO 19111 defines the term “meridian” as the intersection between an ellipsoid and a plane containing the semi-minor axis of the ellipsoid.

[11] ISO 19111 defines the term “prime meridian” as the meridian from which the longitudes of other meridians are quantified. In Clause 7, most, but not all, oblate ellipsoid Earth object reference models associate the Greenwich meridian with the prime meridian (see 7.4.5).

5.6.2 Orthogonality

A CS of CS type 3D, CS type surface, or CS type 2D is *orthogonal* if the angle between any two coordinate-component curves at $\boldsymbol{u}$ is a right angle when $\boldsymbol{u}$ is any coordinate in the interior of the CS domain of the generating function.

EXAMPLE The polar CS of 5.3 EXAMPLE is a orthogonal CS of CS type 2D.

5.6.3 Linear CS properties: Cartesian, and orthonormal

In a linear CS, the k^{th} coordinate-component curve is a line. The k^{th} coordinate-component curve at the origin **0** of a linear CS is the k^{th}-*axis*.

In a linear CS, if the angles between coordinate-component curves at the origin **0** are (pair-wise) right angles, then that is the case at all points. In particular, a linear CS is orthogonal[12] if the axes are orthogonal.

In some publications a Cartesian CS is defined the same way as an orthogonal linear CS[13]. This International Standard, however, defines this concept differently. A linear CS with generating function $\boldsymbol{F}$ is a *Cartesian CS* if $\left\| \boldsymbol{F}(e_i) - \boldsymbol{F}(\mathbf{0}) \right\| = 1, \quad i = 1,\dots,n$ (*i.e.,* the axis unit points are all one unit distant from the origin $\boldsymbol{F}(\mathbf{0})$).

An *orthonormal CS* is a linear CS that is both orthogonal and Cartesian.

A CS of CS type 3D with generating function $\boldsymbol{F}$ is *orientation-preserving* if the Jacobian determinant of $\boldsymbol{F}$ is positive.

EXAMPLE The Lococentric Euclidean 3D CS specified in Table 5.9 is an orientation-preserving orthonormal CS.

5.6.4 CS right-handedness and coordinate-component ordering

Given a CS of CS type 3D and a coordinate $\boldsymbol{c} = (u_0, v_0, w_0)$ in the interior of the CS domain, the coordinate-component curves at $\boldsymbol{p}$ determine an ordered set of three tangent vectors:

$$\boldsymbol{t}_1 = \left(\frac{d\boldsymbol{C}_1}{du} \right)_{u=u_0},$$

$$\boldsymbol{t}_2 = \left(\frac{d\boldsymbol{C}_2}{dv} \right)_{v=v_0}, \text{ and}$$

$$\boldsymbol{t}_3 = \left(\frac{d\boldsymbol{C}_3}{dw} \right)_{w=w_0}.$$

An orthogonal CS of CS type 3D is a *right-handed CS* if for some coordinate $\boldsymbol{c}$ in the interior of the CS domain, the ordered set of tangent vectors $\boldsymbol{t}_1$, $\boldsymbol{t}_2$, and $\boldsymbol{t}_3$ form a right-handed coordinate system as defined in ISO 80000-2. The right-handed CS property is determined, in part, by the order of the coordinate-components in the coordinate 3-tuple. The order of the coordinate-components in the specification of an orthogonal CS of CS type 3D shall be restricted to an ordering that ensures a right-handed CS. This restriction is required for uniform treatment of directions in an SRF (see 10.5).

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[12] Some publications use “rectangular” to denote an orthogonal linear CS, and “oblique” to denote a non-orthogonal linear CS.

[13] ISO 19111 defines “Cartesian coordinate system” as a coordinate system that gives the position of points relative to n mutually-perpendicular axes.

The coordinate-component ordering in the specification of a surface CS that is induced on a coordinate-component surface of a 3D CS, shall use the coordinate-component order of the inducing 3D CS.

EXAMPLE 1 The geodetic 3D CS (Table 5.14) coordinate-component ordering (λ, φ, h) ensures that the CS is right-handed. A similar ordering for the planetodetic 3D CS (Table 5.15) is not right-handed because the tangent to planetodetic longitude points opposite to the direction of the tangent to geodetic longitude. Instead, the coordinate-component ordering (φ, λ, h) is specified to satisfy the right-handed CS requirement.

EXAMPLE 2 The surface planetodetic geodetic CS (Table 5.25) coordinate-component ordering (φ, λ) is determined by the coordinate-component ordering (φ, λ, h) of the planetodetic 3D CS (Table 5.15).

5.7 CS localization

In some applications of a CS in the context of a spatial reference frame, it is necessary to consider a modified version of the CS that has been translated to a local origin and/or has been rotated (see "Lococentric" spatial reference frame variants in Clause 8). To treat these modifications in a uniform manner, the generating function of a CS that has been translated to a local origin and/or has been rotated is related to the generating function of the original CS by means of a localization operator. This uniform method, defined below, of specifying the variant CS by composing the original CS generating function with a localization operator shall be called *CS localization*.

Three parameterized operators, called *localization operators,* that operate on or between position-spaces are defined in Table 5.2. The inverses of these operators are defined in Table 5.3.

Table 5.2 — Localization operators

Localization operator	Domain	Range	Localization parameters	Operator definition
$\boldsymbol{L}_{\text{3D}}$	$\mathbf{R}^3$	$\mathbf{R}^3$	$\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$, in $\mathbf{R}^3$ $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal	$\boldsymbol{L}_{\text{3D}}(x, y, z) = \boldsymbol{q} + x\boldsymbol{r} + y\boldsymbol{s} + z\boldsymbol{t}$, where $\boldsymbol{t} = \boldsymbol{r} \times \boldsymbol{s}$.
$\boldsymbol{L}_{\text{Surface}}$	$\mathbf{R}^2$	$\mathbf{R}^3$	$\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$, in $\mathbf{R}^3$ $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal	$\boldsymbol{L}_{\text{Surface}}(x, y) = \boldsymbol{q} + x\boldsymbol{r} + y\boldsymbol{s}$
$\boldsymbol{L}_{\text{2D}}$	$\mathbf{R}^2$	$\mathbf{R}^2$	$\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$, in $\mathbf{R}^2$ $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal	$\boldsymbol{L}_{\text{2D}}(x, y) = \boldsymbol{q} + x\boldsymbol{r} + y\boldsymbol{s}$

Table 5.3 — Localization inverse operators

Localization operator	Inverse operator definition
$\boldsymbol{L}_{\text{3D}}$	$\boldsymbol{L}_{\text{3D}}^{-1}(\boldsymbol{p}) = ((\boldsymbol{p}-\boldsymbol{q}) \bullet \boldsymbol{r})\boldsymbol{e}_1 + ((\boldsymbol{p}-\boldsymbol{q}) \bullet \boldsymbol{s})\boldsymbol{e}_2 + ((\boldsymbol{p}-\boldsymbol{q}) \bullet \boldsymbol{t})\boldsymbol{e}_3$
$\boldsymbol{L}_{\text{Surface}}$	$\boldsymbol{L}_{\text{Surface}}^{-1}(\boldsymbol{p}) = ((\boldsymbol{p}-\boldsymbol{q}) \bullet \boldsymbol{r})\boldsymbol{e}_1 + ((\boldsymbol{p}-\boldsymbol{q}) \bullet \boldsymbol{s})\boldsymbol{e}_2$
$\boldsymbol{L}_{\text{2D}}$	$\boldsymbol{L}_{\text{2D}}^{-1}(\boldsymbol{p}) = ((\boldsymbol{p}-\boldsymbol{q}) \bullet \boldsymbol{r})\boldsymbol{e}_1 + ((\boldsymbol{p}-\boldsymbol{q}) \bullet \boldsymbol{s})\boldsymbol{e}_2$

There are several forms of CS localization depending on CS type and localization operator. A 3D or surface CS with generating function $\boldsymbol{F}$ is localized by composing $\boldsymbol{F}$ with the $\boldsymbol{L}_{\text{3D}}$ localization operator. The localized CS is of the same CS type (CS type 3D or CS type surface, respectively). Its generating function is $\boldsymbol{F}_{\mathrm{L}} \equiv \boldsymbol{L}_{\mathrm{3D}} \circ \boldsymbol{F}$ and has the same CS domain as $\boldsymbol{F}$.

There are two localization operators for a 2D CS. One uses localization parameters in $\mathbf{R}^3$ and produces a surface CS. The other uses localization parameters in $\mathbf{R}^2$ and produces a 2D CS.

a) A 2D CS with generating function $\boldsymbol{F}$ is localized by composing $\boldsymbol{F}$ with the $\boldsymbol{L}_{\mathrm{Surface}}$ localization operator. The localized CS is a surface CS. Its generating function is $\boldsymbol{F}_{\mathrm{L}} \equiv \boldsymbol{L}_{\mathrm{Surface}} \circ \boldsymbol{F}$ and has the same CS domain as $\boldsymbol{F}$.

b) A 2D CS with generating function $\boldsymbol{F}$ is localized by composing $\boldsymbol{F}$ with the $\boldsymbol{L}_{\mathbf{2D}}$ localization operator. The localized CS is a 2D CS. Its generating function is $\boldsymbol{F}_{\mathrm{L}} \equiv \boldsymbol{L}_{\mathrm{2D}} \circ \boldsymbol{F}$ and has the same CS domain as $\boldsymbol{F}$.

The localization operator parameter $\boldsymbol{q}$ shall be called the *lococentre*. A localized CS may be called a *lococentric* CS.

NOTE CS localization preserves the following CS properties: linear/curvilinear, orthogonal, Cartesian, and orthonormal.

The relationship between a CS type and its localized version(s) is summarized in Table 5.4.

Table 5.4 — Localized CS type relationships

CS type	Localization operator	Lococentric CS type
3D	$\boldsymbol{L}_{\mathrm{3D}}$	3D
Surface	$\boldsymbol{L}_{\mathrm{3D}}$	Surface
2D	$\boldsymbol{L}_{\mathrm{Surface}}$	
	$\boldsymbol{L}_{\mathrm{2D}}$	2D

5.8 Map projection coordinate systems

5.8.1 Map projections

Map projections are 2D models of a 3D curved surface. In this International Standard, map projections are limited to the surface of an oblate ellipsoid. A *map projection* (MP) is comprised of

a) an MP domain in the surface of an oblate ellipsoid,

b) a generating projection, and

c) an MP range in 2D coordinate-space,

where:

a) the MP domain is a connected subset of the surface of the oblate ellipsoid,

b) the MP range is a connected replete set, and

c) the *generating projection* is one-to-one from the MP domain in the oblate ellipsoid onto its MP range and its inverse function is smooth and orientation-preserving in the MP range interior.

NOTE 1 This definition may be generalized to any ellipsoid including tri-axial ellipsoids, but this International Standard only addresses map projections for oblate ellipsoids.

NOTE 2 The domain of a map projection is always a proper subset of the oblate ellipsoid surface. In particular, the domain of the Mercator map projection (see Table 5.18) omits the pole points.

The generating projection $\boldsymbol{P}$ is specified in terms of surface geodetic CS coordinates (see Table 5.24). The component functions P_1 and P_2 of the generating projection $\boldsymbol{P}$ shall be called the *mapping equations*:

$$\boldsymbol{P}(\lambda, \varphi) = (u, v)$$

where:

$u = P_1(\lambda, \varphi)$, and

$v = P_2(\lambda, \varphi)$.

The MP range coordinate-components u and v shall be called *easting* and *northing*, respectively. The positive direction of the u-axis (the easting axis) shall be called *map-east*. The positive direction of the v-axis (the northing axis) shall be called *map-north*.

The inverse mapping equations are the component functions Q_1 and Q_2 of the inverse generating projection $\boldsymbol{Q} = \boldsymbol{P}^{-1}$:

$\lambda = Q_1(u, v)$

$\varphi = Q_2(u, v)$

5.8.2 Map projection as a surface CS

If the inverse generating projection of a map projection $\boldsymbol{Q}$ is composed with the surface geodetic CS generating function $\boldsymbol{G}_{\mathrm{GD}}$, the resulting function $\boldsymbol{G}_{\mathrm{MP}} = \boldsymbol{G}_{\mathrm{GD}} \circ \boldsymbol{Q}$ is the generating function of a surface CS (see Figure 5.4). The CS domain is the MP range. In this International Standard, a *map projection CS* shall be a surface CS for which the generating function is implicitly specified in terms of the mapping equations of a map projection.

In some cases, the surface geodetic coordinates with coordinate-component $\varphi = \pm\pi/2$ are not in the MP domain of $\boldsymbol{P}$ nor are they in the range of $\boldsymbol{Q}$. However, if the composite function $\boldsymbol{G}_{\mathrm{MP}}^{-1} = \boldsymbol{P} \circ \boldsymbol{G}_{\mathrm{GD}}^{-1}$ is continuous at the pole points $(0, 0, \pm b)$, then $\boldsymbol{G}_{\mathrm{MP}}$ and $\boldsymbol{G}_{\mathrm{MP}}^{-1}$ shall be extended by continuity to include the pole points in the CS range.

NOTE The CS generating function $\boldsymbol{G}_{\mathrm{MP}} = \boldsymbol{G}_{\mathrm{GD}} \circ \boldsymbol{Q}$ is not to be confused with the generating projection $\boldsymbol{P}$.

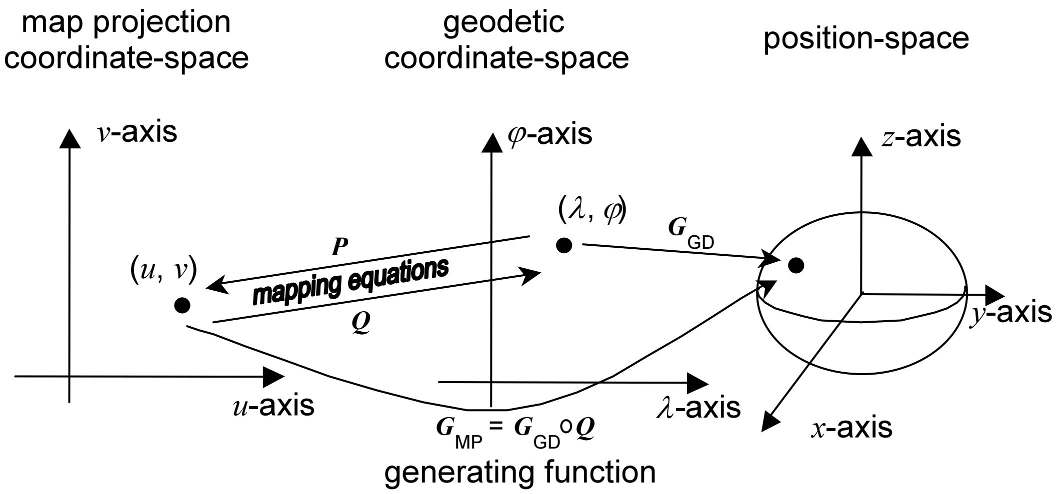

Figure 5.4 — The generating function of a map projection

5.8.3 Map projection geometry

5.8.3.1 Introduction

In general, the Euclidean geometry that a surface CS 2D coordinate-space inherits from $\mathbf{R}^2$ has no direct significance with respect to the geometry of position-space. In particular, the Euclidean distance between a pair of surface geodetic coordinates has no obvious meaning in position-space. In contrast, map projections are specifically designed so that coordinate-space geometry will model one or more geometric aspects of the corresponding oblate ellipsoid surface in position-space.

The map projection CSs specified in this International Standard are designed so that one or more geometric aspects of the MP domain in the oblate ellipsoid surface are approximated or modelled by the corresponding aspect in coordinate-space. The length of the line segment between two map coordinates is related to the length of the corresponding surface curve. Similarly, one or more of directions, areas, the angles between two intersecting curves, and shapes may be related approximately or exactly to the corresponding geometric aspect on the oblate ellipsoid surface.

The extent to which these aspects are or are not closely related is an indication of distortion. Some map projection CSs are designed to eliminate distortion for one geometric aspect (such as angles or area). Others are designed to reduce distortion for several geometric aspects. In general, distortion tends to increase with the size of the oblate ellipsoid MP domain relative to the total oblate ellipsoid surface area. Conversely, distortion errors may be reduced by restricting the size of the MP domain. Map projections specified in this International Standard in the context of a spatial reference frame may have areas of definition beyond which the projection should not be used for some application domains due to unacceptable distortion[14].

5.8.3.2 Conformal map projections

A *conformal map projection* preserves angles. For such map projections, when two surface curves on an oblate ellipsoid meet at the angle α, the image of those curves in the map coordinate-space meet at the same angle α [THOM].

[14] It is a consequence of the *Theorema Egregium* of Gauss that no map projection CS can eliminate all distortion.

In addition, [THOM] contains a derivation based on the theory of complex variables to obtain conditions that specify when a projection is conformal. The map projections specified in Table 5.18 through Table 5.22 are conformal. The equidistant cylindrical MP specified in Table 5.23 is not conformal.

NOTE The conformal property is local. A conformal map projection preserves angles at a point, but does not necessarily preserve shape or area. In particular, a large projected triangle may appear distorted under a conformal map projection.

5.8.3.3 Point distortion

One indicator of map projection length distortion is the ratio of lengths between an infinitesimal line segment in coordinate-space and the corresponding curve in position-space. Given a point in the interior of the MP range with surface geodetic coordinate (λ,φ) the *directional point distortion*[15] at (λ,φ) with respect to a smooth surface curve passing through the point is the ratio of the differential distance in coordinate-space to the differential arc length at (λ,φ) along the curve as determined by the mapping equations.

The *latitudinal point distortion* at (λ,φ), denoted $j(\lambda,\varphi)$, is the directional point distortion with respect to the meridian at (λ,φ). It is computed in the direction of the meridian at the point as:

$$j(\lambda,\varphi) = \lim_{\Delta\to 0}\frac{\Delta(\text{arc length in coordinate-space})}{\Delta(\text{arc length along a meridian})} = \frac{\sqrt{(\partial u/\partial\varphi)^2 + (\partial v/\partial\varphi)^2}}{R_M(\varphi)}$$

where $R_M(\varphi)$ is the radius of curvature in the meridian as specified in Table 5.6.

The *longitudinal point distortion* at (λ,φ), denoted $k(\lambda,\varphi)$, is the directional point distortion with respect to the parallel at (λ,φ). It is computed in the direction of the parallel at the point as:

$$k(\lambda,\varphi) = \lim_{\Delta\to 0}\frac{\Delta(\text{arc length in coordinate-space})}{\Delta(\text{arc length along a parallel})} = \frac{\sqrt{(\partial u/\partial\lambda)^2 + (\partial v/\partial\lambda)^2}}{R_N(\varphi)\cos(\varphi)}$$

where $R_N(\varphi)$ is the radius of curvature in the prime vertical as specified in Table 5.6.

If a map projection is conformal, then the directional point distortion is independent of the direction of the curve at the point. In particular, $j(\lambda,\varphi) = k(\lambda,\varphi)$ for conformal map projections.

It is common practice in cartography to convert map projection coordinate-space to a display coordinate-space by means of a scaling factor. The scaling factor σ shall be termed a *map scale* [HTDP] and a point in the display space shall be termed a *display coordinate*[16]. The relationship of a display coordinate (u_d, v_d) to a map coordinate (u, v) is:

$$u_d = \sigma u$$
$$v_d = \sigma v$$

Map scale is commonly expressed as a ratio 1:n.

EXAMPLE A map scale printed on a map sheet as 1:50 000 corresponds to σ = 1/50 000.

15 This concept is found in the literature under a variety of names. The term “point distortion” is introduced to avoid ambiguity.

16 The distinction between a map projection coordinate and a display coordinate is not usually made explicit in the literature. The term “display coordinate” is introduced to avoid ambiguity.

For a conformal map projection, the infinitesimal ratio of display distance to arc length along a parallel is the *point scale* at (λ, φ) and is denoted by k_{scaled}. The relationship between point scale and point distortion is:

$$k_{\text{scaled}}(\lambda, \varphi) = \sigma\, k(\lambda, \varphi).$$

5.8.3.4 Geodetic azimuth and map azimuth

The *geodetic azimuth*[17] from a non-polar point $\boldsymbol{p}_1$ on the surface of an ellipsoid to a second point $\boldsymbol{p}_2$ on the surface is the angle measured clockwise from the meridian curve segment connecting $\boldsymbol{p}_1$ to the North pole to the geodesic containing $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ (see Figure 5.5). The range of azimuth values α shall be $0 \le \alpha < 2\pi$. The definition and range constraints apply to points in both hemispheres.

In a map projection CS, the *map azimuth* from a coordinate $\boldsymbol{c}_1$ to a coordinate $\boldsymbol{c}_2$ is defined as the angle from the v-axis (map-north) clockwise to the line segment connecting $\boldsymbol{c}_1$ to $\boldsymbol{c}_2$. In general, the map azimuth for a pair of coordinates will differ in value from the geodetic azimuth of the corresponding points on the oblate ellipsoid.

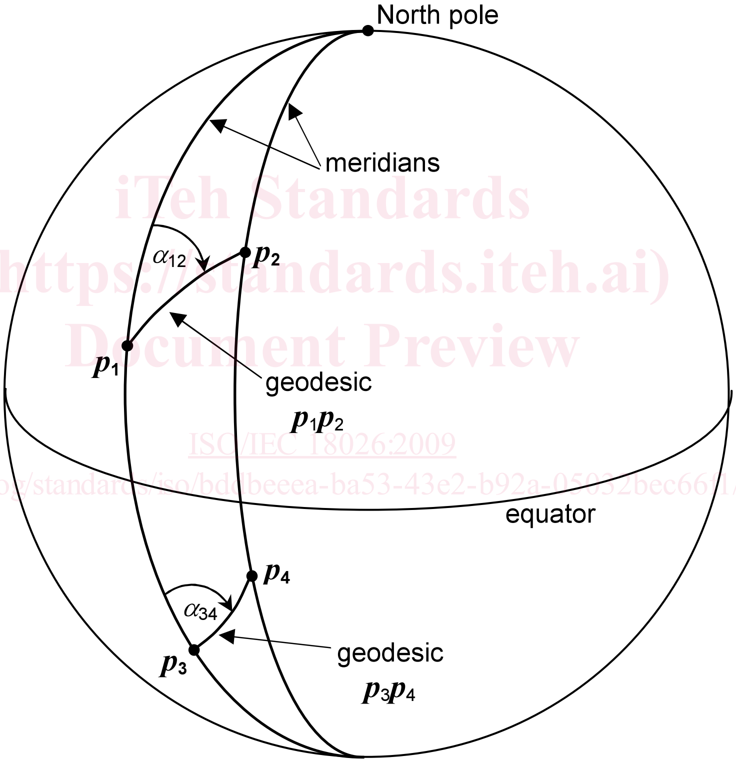

Figure 5.5 — Geodetic azimuths α_{12} from p_1 to p_2 and α_{34} from p_3 to p_4

[17] More general definitions that allow measurements of azimuth angle clockwise or counter-clockwise and from the north or south side of the meridian are in use. The generalization to the case for which one or more of the two points is not on the surface is treated in [RAPP1] and [RAPP2]. The more general definitions are not required for subsequent SRM concepts.

5.8.3.5 Convergence of the meridian

Given a point (λ, φ) in the interior of the MP domain of a map projection, the meridian through that point is projected to a curve in coordinate-space that passes through the corresponding coordinate. The angle γ at the coordinate in the clockwise direction from the curve to the v-axis (map-north) direction shall be called the *convergence of the meridian* (COM) (see Figure 5.6).

The relationship $\gamma(\lambda, \varphi) = \text{arctan2}\left(-\dfrac{\partial u}{\partial \varphi}, \dfrac{\partial v}{\partial \varphi}\right)$ is used to derive the formulae for COM from the mapping equations of each of the map projections[18]. The COM angle is adjusted to the range $-\pi < \gamma \leq \pi$.

NOTE If the map projection is conformal, then an equivalent relationship is given by: $\gamma(\lambda, \varphi) = \text{arctan2}\left(\dfrac{\partial v}{\partial \lambda}, \dfrac{\partial u}{\partial \lambda}\right)$.

A typical geometry illustrating the COM at a point $\boldsymbol{p}$ is shown for the transverse Mercator map projection in Figure 5.6.

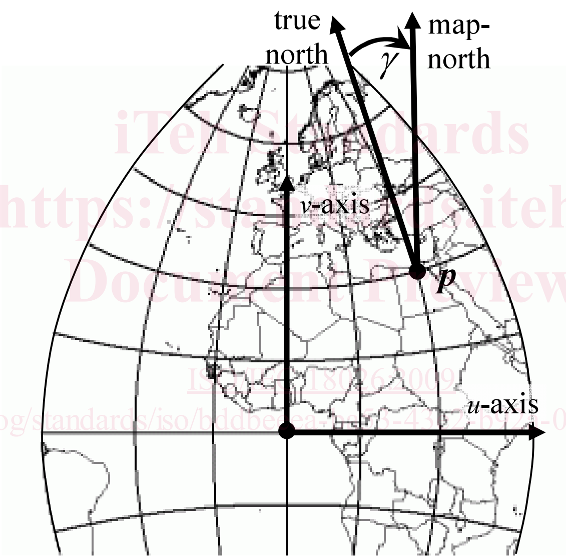

Figure 5.6 — Convergence of the meridian

EXAMPLE If $\boldsymbol{p}_2$ is directly map-north of $\boldsymbol{p}_1$ (it has a larger v coordinate-component), then the map azimuth is zero, but the geodetic azimuth may not be zero. The geodetic azimuth is approximately the sum of the map azimuth and the COM if the points are sufficiently close together.

5.8.4 Relationship to projection functions

5.8.4.1 Projection functions

Projection functions are defined in A.9. In some cases, the generating projection of a map projection CS is derived from a projection function. The derivation involves two steps. The first step is to restrict the domain of

[18] The function arctan2 is defined in A.8.2.

