Annex A
(normative)

Mathematical foundations

A.1 Introduction

This annex identifies the concepts from mathematics used in this International Standard and specifies the notation used for those concepts. No proofs are presented. A reader of this International Standard is assumed to be familiar with mathematics including set theory, linear algebra, and the calculus of several real variables as presented in reference works such as the *Encyclopedic Dictionary of Mathematics* [EDM].

A.2 $\mathbf{R}^n$ as a real vector space

An ordered set of n real numbers $\boldsymbol{a}$ where n is a natural number is called an *n-tuple of real numbers* and shall be denoted by $\boldsymbol{a} = (a_1, a_2, a_3, \cdots, a_n)$ The set of all n-tuples of real numbers is denoted by $\mathbf{R}^n$. $\mathbf{R}^n$ is an n-dimensional vector space.

The *canonical basis* for $\mathbf{R}^n$ is defined as:

$$e_1 = (1, 0, \cdots, 0),\ e_2 = (0, 1, \cdots, 0), \cdots,\ e_n = (0, 0, \cdots, 1).$$

The elements of $\mathbf{R}^n$ may be called *points* or *vectors*. The latter term is used in the context of directions or vector space operations.

The zero vector $(0, 0, \cdots, 0)$ is denoted by **0**.

Definitions A.2(a) through A.2(j) apply to any vectors $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$ in $\mathbf{R}^n$:

a) The *inner product* or *dot-product* of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as:

$$\boldsymbol{x} \bullet \boldsymbol{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n .$$

b) Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are called *orthogonal* if $\boldsymbol{x} \bullet \boldsymbol{y} = 0$.

c) If $n \geq 2$, two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are called *perpendicular* if and only if they are orthogonal.

NOTE 1 If $n \geq 2$, $\boldsymbol{x} \bullet \boldsymbol{y} = \|\boldsymbol{x}\| \|\boldsymbol{y}\| \cos(\alpha)$ where α is the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$.

d) $\boldsymbol{x}$ is called *orthogonal to a set* of vectors if $\boldsymbol{x}$ is orthogonal to each vector that is a member of the set.

e) The *norm* of $\boldsymbol{x}$ is defined as

$$\| \boldsymbol{x} \| = \sqrt{\boldsymbol{x} \bullet \boldsymbol{x}} .$$

NOTE 2 The norm of $\boldsymbol{x}$ represents the length of the vector $\boldsymbol{x}$. Only the zero vector **0** has norm zero.

f) $\boldsymbol{x}$ is called a *unit vector* if $\|\boldsymbol{x}\| = 1$.

g) A set of two or more orthogonal unit vectors is called an *orthonormal set of vectors*.

EXAMPLE The canonical basis is an example of an orthonormal set of vectors.

h) The *Euclidean metric* d is defined by

$$d(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}||.$$

i) The value of $d(\boldsymbol{x}, \boldsymbol{y})$ is called the *Euclidean distance* between $\boldsymbol{x}$ and $\boldsymbol{y}$.

j) The *cross product* of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbf{R}^3$ is defined as the vector:

$$\boldsymbol{x} \times \boldsymbol{y} = \left(x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1\right).$$

NOTE 3 The vector $\boldsymbol{x} \times \boldsymbol{y}$ is orthogonal to both $\boldsymbol{x}$ and $\boldsymbol{y}$, and

$$\|\boldsymbol{x} \times \boldsymbol{y}\| = \|\boldsymbol{x}\|\|\boldsymbol{y}\| \sin(\alpha),$$

where α is the angle between vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

A.3 The point set topology of $\mathbf{R}^n$

Given a point $\boldsymbol{p}$ in $\mathbf{R}^n$ and a real value $\varepsilon > 0$, the set $\{\boldsymbol{q}$ in $\mathbf{R}^n \mid d(\boldsymbol{p}, \boldsymbol{q}) < \varepsilon\}$ is called the *ε-neighbourhood* of $\boldsymbol{p}$.

Given a set $D \subset \mathbf{R}^n$ and a point $\boldsymbol{p}$, the following terms are defined:

a) $\boldsymbol{p}$ is an *interior point of* D if at least one ε-neighbourhood of $\boldsymbol{p}$ is a subset of D.

b) The *interior of a set* D is the set of all points that are interior points of D.

NOTE 1 The interior of a set may be empty.

c) D is *open* if each point of D is an interior point of D. Consequently, D is open if it is equal to its interior.

d) $\boldsymbol{p}$ is a *closure point of* D if every ε-neighbourhood of $\boldsymbol{p}$ has a non-empty intersection with D.

NOTE 2 Every member of D is a closure point of D.

e) The *closure of a set* D is the set of all points that are closure points of D.

f) D is a *closed set* if it is equal to the closure set of D.

g) A set D is *replete* if all points in D belong to the closure of the interior of D.

NOTE 3 Every open set is replete. The union of an open set with any or all of its closure points forms a replete set. In particular, the closure of an open set is replete.

EXAMPLE 1 In $\mathbf{R}^2$ $\{(x, y) \mid -\pi < x < \pi, -\pi/2 < y < \pi/2\}$ is open and therefore replete.

EXAMPLE 2 $\{(x, y) \mid -\pi < x \le \pi, -\pi/2 < y < \pi/2\}$ is replete.

EXAMPLE 3 $\{(x, y) \mid -\pi \le x \le \pi, -\pi/2 \le y \le \pi/2\}$ is closed and replete.

A.4 Smooth functions on $\mathbf{R}^n$

A real-valued function f defined on a replete domain in $\mathbf{R}^n$ is called *smooth* if its first derivative exists and is continuous at each point in its domain.

The *gradient* of f is the vector of first order partial derivatives

$$\mathbf{grad}(f) = \left(\frac{\partial f}{\partial v_1}, \frac{\partial f}{\partial v_2}, \cdots, \frac{\partial f}{\partial v_n}\right).$$

Definitions A.4(a) through A.4(g) apply to any vector-valued function $\boldsymbol{F}$ defined on a replete domain D in $\mathbf{R}^n$ with range in $\mathbf{R}^m$.

a) The j^{th}-*component function* of a vector-valued function $\boldsymbol{F}$ is the real-valued function f_j defined by $f_j = \boldsymbol{e}_j \bullet \boldsymbol{F}$ where $\boldsymbol{e}_j$ is the j^{th} canonical basis vector, $j = 1, 2, \cdots, m$.

In this case:

$\boldsymbol{F}(\boldsymbol{v}) = (f_1(\boldsymbol{v}), f_2(\boldsymbol{v}), f_3(\boldsymbol{v}), \ldots, f_m(\boldsymbol{v}))$ for $\boldsymbol{v} = (v_1, v_2, v_3, \ldots, v_n)$ in D.

b) $\boldsymbol{F}$ is called *smooth* if each component function fj is smooth.

c) The *first derivative* of a smooth vector-valued function $\boldsymbol{F}$, denoted $\mathbf{d}\boldsymbol{F}$, evaluated at a point in the domain is the $n \times m$ matrix of partial derivatives evaluated at the point:

$$\left(\frac{\partial f_j}{\partial v_i}\right) \quad i = 1, 2, \cdots, n \text{ and } j = 1, 2, \cdots, m .$$

d) The *Jacobian matrix* of $\boldsymbol{F}$ at the point $\boldsymbol{v}$ is the matrix of the first derivative of $\boldsymbol{F}$.

NOTE 1 The rows of the Jacobian matrix are the gradients of the component functions of $\boldsymbol{F}$.

e) In the case $m = n$, the Jacobian matrix is square and its determinant is called the *Jacobian determinant.*

f) In the case $m = n$, $\boldsymbol{F}$ is called *orientation preserving* if its Jacobian determinant is strictly positive for all points in D.

g) A vector-valued function $\boldsymbol{F}$ defined on $\mathbf{R}^n$ is *linear* if:

$\boldsymbol{F}(a\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{F}(\boldsymbol{y})$ for all real scalars a and vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbf{R}^n$.

NOTE 2 All linear functions are smooth.

A vector-valued function $\boldsymbol{E}$ defined on $\mathbf{R}^n$ is *affine* if $\boldsymbol{F}$, defined by $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{E}(\boldsymbol{x}) - \boldsymbol{E}(\mathbf{0})$, is a linear function. All affine functions on $\mathbf{R}^n$ are smooth.

A function may be alternatively called an *operator* especially when attention is focused on how the function maps a set of points in its domain onto a corresponding set of points in its range.

EXAMPLE The localization operators (see 5.7).

A.5 Functional composition

If $\boldsymbol{F}$ and $\boldsymbol{G}$ are two vector valued functions and the range of $\boldsymbol{G}$ is contained in the domain of $\boldsymbol{F}$, then $\boldsymbol{F} \circ \boldsymbol{G}$, the *composition* of $\boldsymbol{F}$ with $\boldsymbol{G}$, is the function defined by $\boldsymbol{F} \circ \boldsymbol{G}(\boldsymbol{x}) \equiv \boldsymbol{F}(\boldsymbol{G}(\boldsymbol{x}))$. $\boldsymbol{F} \circ \boldsymbol{G}$ has the same domain as $\boldsymbol{G}$, and the range of $\boldsymbol{F} \circ \boldsymbol{G}$ is contained in the range of $\boldsymbol{F}$.

Functional composition also applies to scalar-valued functions f and g, If the range of g is contained in the domain of f, then $f \circ g(x)$, the composition of f with g, is the function defined by $f \circ g(x) \equiv f(g(x))$.

A.6 Smooth surfaces in $\mathbf{R}^3$

A.6.1 Implicit definition

A *smooth surface* in $\mathbf{R}^3$ is *implicitly* specified by a real-valued smooth function f defined on $\mathbf{R}^3$ as the set S of all points (x, y, z) in $\mathbf{R}^3$ satisfying:

a) $f(x, y, z) = 0$ and

b) $\mathbf{grad}(f)(x, y, z) \neq \mathbf{0}$.

In this case, f is called a *surface generating function* for the surface S.

EXAMPLE 1 If $\boldsymbol{n} \neq \mathbf{0}$ and $\boldsymbol{p}$ are vectors in $\mathbf{R}^3$ and $f(\boldsymbol{v}) = \boldsymbol{n} \bullet (\boldsymbol{v} - \boldsymbol{p})$, then f is smooth and $\mathbf{grad}(f) = \boldsymbol{n} \neq \mathbf{0}$. The plane which is perpendicular to $\boldsymbol{n}$ and contains $\boldsymbol{p}$ is the smooth surface implicitly defined by the surface generating function f.

Special cases:

When $\boldsymbol{n} = (1, 0, 0)$ and $\boldsymbol{p} = \mathbf{0}$, the yz-plane is implicitly defined.

When $\boldsymbol{n} = (0, 1, 0)$ and $\boldsymbol{p} = \mathbf{0}$, the xz-plane is implicitly defined.

When $\boldsymbol{n} = (0, 0, 1)$ and $\boldsymbol{p} = \mathbf{0}$, the xy-plane is implicitly defined.

The *surface normal* $\boldsymbol{n}$ at a point $\boldsymbol{p} = (x, y, z)$ on the surface implicitly specified by a surface generating function f is defined as:

$$\boldsymbol{n} \equiv \frac{1}{\left\|\mathbf{grad}(f)(\boldsymbol{p})\right\|}\mathbf{grad}(f)(\boldsymbol{p}).$$

NOTE $-\boldsymbol{n}$ is also a surface normal to S at $\boldsymbol{p}$. The surface generating function f determines the surface normal direction: $\boldsymbol{n}$ or $-\boldsymbol{n}$.

The *tangent plane* to a surface at a point $\boldsymbol{p} = (x, y, z)$ on the surface S implicitly defined by a surface generating function f is the plane which is the smooth surface implicitly defined by $h(\boldsymbol{v}) = \boldsymbol{n} \bullet (\boldsymbol{v} - \boldsymbol{p})$ where $\boldsymbol{n}$ is the surface normal to S at $\boldsymbol{p}$.

EXAMPLE 2 If a and b are positive non-zero scalars, define

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} - 1.$$

Then f is smooth and

$$\mathbf{grad}(f)(x, y, z) = \left(\frac{2x}{a^2}, \frac{2y}{a^2}, \frac{2z}{b^2}\right)$$

is never (0, 0, 0) on the surface implicitly specified by the set satisfying $f = 0$.

A.6.2 Ellipsoid surfaces

If a and b are positive non-zero scalars, the smooth function:

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} - 1$$

is a surface generating function for an *ellipsoid of revolution* smooth surface S.

When $b \le a$, the surface is called an *oblate ellipsoid*. In this case a is called the *major semi-axis*[29] of the oblate ellipsoid and b is called the *minor semi-axis* of the oblate ellipsoid.

The *flattening* of an oblate ellipsoid is defined as $f = (a - b)/a$.

The *eccentricity* of an oblate ellipsoid is defined as $\varepsilon = \sqrt{1-(b/a)^2}$.

The *second eccentricity* of an oblate ellipsoid is defined as $\varepsilon' = \sqrt{(a/b)^2 - 1}$.

When $b = a$, the oblate ellipsoid may be called a *sphere* of radius $r = b = a$.

When $a < b$, the surface is called a *prolate ellipsoid*. In this case, a is called the *minor semi-axis* of the prolate ellipsoid and b is called the *major semi-axis* of the prolate ellipsoid.

NOTE 1 A sphere of radius r is also implicitly defined by the surface generating function $f(x, y, z) = x^2 + y^2 + z^2 - r^2$.

NOTE 2 The term spheroid is often used to denote an oblate ellipsoid with an eccentricity close to zero ("almost spherical").

A.7 Smooth curves in $\mathbf{R}^n$

A.7.1 Parametric definition

A.7.1.1 Smooth curve

A *smooth curve* in $\mathbf{R}^n$ is *parametrically* specified by a smooth one-to-one $\mathbf{R}^n$ valued function $\boldsymbol{F}(t)$ defined on a replete interval I in $\mathbf{R}$ such that $\|\mathbf{d}\boldsymbol{F}(t)\| \ne 0$ for any t in I.

EXAMPLE 1 If $\boldsymbol{p}$ and $\boldsymbol{n}$ are vectors in $\mathbf{R}^n$ such that $\boldsymbol{n} \ne \mathbf{0}$ and $\boldsymbol{L}(t) = \boldsymbol{p} + t\,\boldsymbol{n}$, $-\infty < t < +\infty$, then $\boldsymbol{L}$ is smooth and $\|\mathbf{d}\boldsymbol{L}(t)\| = \|\boldsymbol{n}\| > 0$. The line which is parallel to $\boldsymbol{n}$ and which contains $\boldsymbol{p}$ is a smooth curve parametrically specified by $\boldsymbol{L}$.

EXAMPLE 2 If a and b are positive non-zero scalars and $b \le a$, define

$$\boldsymbol{F}(t) = (a\cos(t),\ b\sin(t)) \text{ for all } t \text{ in the interval } -\pi < t \le \pi.$$

Then $\boldsymbol{F}$ is smooth and $\|\mathbf{d}\boldsymbol{F}(t)\| \ge b > 0$ for all t in the interval and therefore parametrically specifies a smooth curve in $\mathbf{R}^2$.

An *ellipse* in $\mathbf{R}^2$ with major semi-axis a and minor semi-axis b, $0 < b \le a$, is parametrically specified by:

$$\boldsymbol{F}(t) = (a\cos(t),\ b\sin(t)), \text{ for all } t \text{ in the interval } -\pi < t \le \pi.$$

A.7.1.2 Tangent to a smooth curve

If $\boldsymbol{C}(t)$ parametrically specifies a smooth curve C passing through a point $\boldsymbol{p} = \boldsymbol{C}(t_p)$, the *tangent vector* to C at $\boldsymbol{p}$ shall be defined as:

$$\boldsymbol{t} = \frac{1}{\left\|\mathbf{d}\boldsymbol{C}(t_p)\right\|}\mathbf{d}\boldsymbol{C}(t_p)$$

where $\mathbf{d}\boldsymbol{C}(t_p) = (\mathrm{d}C_1/\mathrm{d}t,\ \mathrm{d}C_2/\mathrm{d}t,\ \ldots,\ \mathrm{d}C_n/\mathrm{d}t)$ is the first derivative of $\boldsymbol{C}$ evaluated at t_p.

NOTE $-\boldsymbol{t}$ is also a tangent vector to C at $\boldsymbol{p}$. The parameterization function $\boldsymbol{C}(t)$ determines the tangent vector direction: $\boldsymbol{t}$ or $-\boldsymbol{t}$.

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29 a is half the length of the major axis. ISO 19111 labels the symbol a as the semi-major axis.

A locus of points is a *directed curve* if it is the range of a smooth curve.

The *tangent line* to the curve C at $\boldsymbol{p}$ is a smooth curve parametrically specified by $\boldsymbol{T}(s) = \boldsymbol{p} + s\,\boldsymbol{t}$, $-\infty < s < +\infty$, where $\boldsymbol{t}$ is a tangent vector to C at $\boldsymbol{p}$. See Figure A.1.

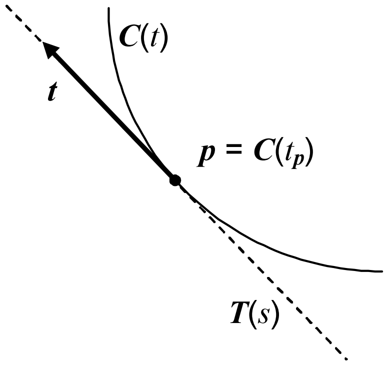

Figure A.1 — Tangent to a curve

A.7.1.3 Angle between curves

If two parametrically specified smooth curves C_1 and C_2 intersect at a point $\boldsymbol{p}$ then the *angle at* $\boldsymbol{p}$ *from* C_1 *to* C_2 is defined as the angle from the tangent vector $\boldsymbol{t}_1$ to the tangent vector $\boldsymbol{t}_2$ of the two curves, respectively, at $\boldsymbol{p}$. This is illustrated in Figure A.2.

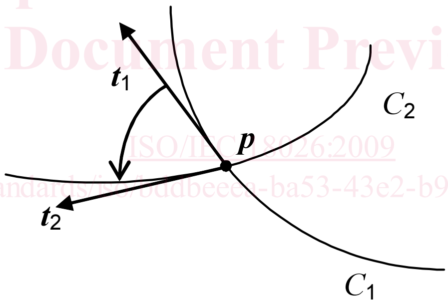

Figure A.2 — Angle between two curves

A.7.1.4 Closed curve

If a smooth function $\boldsymbol{F}$ is defined on a closed and bounded interval I with interval end points t_0 and t_1 and if $\boldsymbol{F}$ parametrically specifies a smooth curve on the interior of I and $\boldsymbol{p} = \boldsymbol{F}(t_0) = \boldsymbol{F}(t_1)$, then $\boldsymbol{F}$ generates a *closed curve* through $\boldsymbol{p}$.

EXAMPLE

$$\boldsymbol{F}(t) = (a\cos(t),\ b\sin(t)),\text{ for all } t \text{ in the interval } -\pi+\theta \le t \le \pi+\theta.$$

If a and b are positive non-zero scalars and θ is given, $\boldsymbol{F}$ generates a closed curve though $\boldsymbol{p} = (a\cos(\pi+\theta),\ b\sin(\pi+\theta))$

