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Speech and multimedia Transmission Quality (STQ); QoS aspects for popular services in mobile networks; Part 6: Post processing and statistical methods

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Foreword

This Technical Specification (TS) has been produced by ETSI Technical Committee Speech and multimedia Transmission Quality (STQ).

The present document is part 6 of a multi-part deliverable. Full details of the entire series can be found in part 1 [i.2].

Modal verbs terminology

In the present document "**shall**", "**shall not**", "**should**", "**should not**", "**may**", "**need not**", "**will**", "**will not**", "**can**" and "**cannot**" are to be interpreted as described in clause 3.2 of the [ETSI Drafting Rules](#) (Verbal forms for the expression of provisions).

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Introduction

All the defined quality of service parameters and their computations are based on field measurements. That indicates that the measurements were made from users point of view (full end-to-end perspective, taking into account the needs of testing).

It is assumed that the end user can handle his mobile and the services he wants to use (operability is not evaluated at this time). For the purpose of measurement it is assumed:

- that the service is available and not barred for any reason;
- routing is defined correctly without errors; and
- the target subscriber equipment is ready to answer the call.

Speech quality values measured should only be employed by calls ended successfully for statistical analysis.

However, measured values from calls ended unsuccessfully (e.g. dropped) should be available for additional evaluations and therefore, need to be stored.

Further preconditions may apply when reasonable.

1 Scope

The present document describes definitions and procedures to be used for statistical calculations which are related to Quality of Service (QoS) measurements done by serving probing systems in mobile communications networks, especially GSM and 3G networks. Network performance measurements and their related post-processing are only marginally covered in the present document.

2 References

2.1 Normative references

References are either specific (identified by date of publication and/or edition number or version number) or non-specific. For specific references, only the cited version applies. For non-specific references, the latest version of the referenced document (including any amendments) applies.

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NOTE: While any hyperlinks included in this clause were valid at the time of publication, ETSI cannot guarantee their long term validity.

The following referenced documents are not necessary for the application of the present document but they assist the user with regard to a particular subject area.

[i.1] ETSI EG 201 769: "Speech Processing, Transmission and Quality Aspects (STQ); QoS parameter definitions and measurements; Parameters for voice telephony service required under the ONP Voice Telephony Directive 98/10/EC".

[i.2] ETSI TS 102 250-1: "Speech and multimedia Transmission Quality (STQ); QoS aspects for popular services in mobile networks; Part 1: Assessment of Quality of Service".

[i.3] NIST/SEMATECH: "e-Handbook of Statistical Methods".

NOTE: Available at <http://www.itl.nist.gov/div898/handbook/>, retrieved 17 September 2019.

[i.4] A. M. Law, W. D. Kelton: "Simulation modeling and analysis", McGraw-Hill, 3rd edition, 2000.

[i.5] J. Hartung: "Lehr- und Handbuch der angewandten Statistik", Oldenbourg Wissenschaftsverlag, 13th meditation, 2002.

[i.6] Bates, D.M. and Chambers, J.M: "Nonlinear regression Analysis and Applications", Wiley & Sons, 1988.

[i.7] Mood, Graybill, Boes: "Introduction to the theory of statistics", McCraw-Hill Statistics Series, 1974.

[i.8] Venables, W.N. and Ripley, B.D.: "Modern Applied Statistics with S-Plus", Springer Verlag, 1999.

3 Definition of terms, symbols and abbreviations

3.1 Terms

For the purposes of the present document, the following terms apply:

rate: measurement result which is related to the portion of time during which it has been executed

NOTE: The denominator's unit is related to time.

ratio: measurement result which quantifies how a subgroup of all single measurements is related to the total number of executed single measurements

NOTE: Usually, nominator and denominator share the same unit, namely a counter for measurements (subgroup/all).

3.2 Symbols

For the purposes of the present document, the following symbols apply:

$E(x)=\mu$	Expected value of random variable x
$Var(x)=\sigma^2$	Variance of random variable x
σ	Standard deviation of random variable x
$f(x)$	Probability Density Function (PDF) of random variable x
$F(x)$	Cumulative Distribution Function (CDF) of random variable x
$S, x \in S$	Set of discrete values or interval of values the random variable x may take
IR	Set of real numbers
s, s^2	Empirical standard deviation / variance, analogous to σ and σ^2 (theoretical)
q_α	α -Quantile
u_α	α -Quantile of standard normal distribution
$x_{(i)}, x_{(1)}, x_{(n)}$	i -th ordered value, minimum and maximum of a given data set $x_i, i = 1, \dots, n$

3.3 Abbreviations

For the purposes of the present document, the following abbreviations apply:

3G	Third Generation
ARMA	Auto-Regressive Moving Average
AVGn	Averaging Operator (regarding n days)
BH	Busy Hour
BSC	Base Station Controller
CDF	Cumulative Distribution Function or Cumulative Density Function
CUSUM	Cumulated SUM
EWMA	Exponentially Weighted Moving Average
GSM	Global System for Mobile communications
KPI	Key Performance Indicator
LSL	Lower Specification Level
MAWD	Monthly Average Working Day
MMQ-Plot	Median-Mean-Quantile Plot
MMS	Multimedia Messaging Service
MOS	Mean Opinion Score
MSC	Mobile Switching Centre
NE	Network Element
PDF	Probability Distribution Function or Probability Density Function

QoS	Quality of Service
QQ-Plot	Quantile-Quantile Plot
SMS	Short Message Service
USL	Upper Specification Level

4 Important measurement data types in mobile communications

4.1 Data with binary values

Appropriate data analysis methods should depend on the type of the given data as well as on the scope of the analysis. Therefore before analysis methods are described, different data types are introduced and differences between them are pointed out.

Four general categories of measurement results are expected when QoS measurements are done in mobile communications.

Single measurements related to the data with binary values:

- service accessibility, service availability;
- service retainability, service continuity;
- error ratios, error probabilities;

in general show a binary outcome, i.e. only two outcomes are possible. This means the result of a single trial leads to a result which is either valued positive or negative related to the considered objective. The result may be recorded as decision-results Yes / No or True / False or with numerical values 0 = successful and 1 = unsuccessful (i.e. errors occur) or vice versa. Aggregation of trials of both types allows to calculate the according ratios which means the number of positive / negative results is divided by the number of all trials. Usually, the units of nominator and denominator are the same, namely number of trials.

EXAMPLE: If established speech calls are considered to test the service retainability of a speech telephony system, every successfully completed call leads to the positive result "Call completed", every unsuccessfully ended call is noticed as "Dropped call" which represents the negative outcome. After 10 000 established calls, the ratio of dropped calls related to all established calls can be calculated. The result is the call drop probability.

4.2 Data out of time-interval measurements

Measurements related to the time domain occur in the areas:

- duration of a session or call;
- service access delay;
- round trip time and end-to-end delay of a service;
- blocking times, downtimes of a system.

The outcome of such measurements is the time span between two time stamps marking the starting and end point of the time periods of interest. Results are related to the unit "second" or multiples or parts of it. Depending on the measurement tools and the precision needed, arbitrarily small measurement units may be realized.

EXAMPLE: Someone can define the end-to-end delivery time for the MMS service by a measurement which starts when the user at the A party pushes the "Send" button and which stops when the completely received MMS is signalled to the user at the B party.

4.3 Measurement of data throughput

Measurements related to data throughput result in values which describe the ratio of transmitted data volume related to the required portion of time. The outcome of a single measurement is the quotient of both measures. Used units are "bit" or multiples thereof for the data amount and "second" or multiples or parts thereof for the portion of time.

EXAMPLE: If a data amount of 1 Mbit is transmitted within a period of 60 seconds, this results in a mean data rate of approximately 16,66 kbit/s.

4.4 Data concerning quality measures

Examples are given by the quality of data transfer which may be measured by its speed or evaluations of speech quality measured on a scale, respectively.

Measurements related to audio-visual quality can be done objectively by algorithms or subjectively by human listeners. The outcome of audio-visual quality evaluation is related to a scaled value which is called Mean Opinion Score (MOS) for subjective testing. Thereby two types of quality measurement are distinguished subjective and objective measurements. If quantitative measures are identified which are highly correlated to the quality of interest, this will simplify the analysis. However, if this is not possible, some kind of evaluation on a standardized scale by qualified experts is needed. The result may therefore be given either as the measurement result or as a mark on a pre-defined scale.

EXAMPLE: Within a subjective test, people are asked to rate the overall quality of video samples which are presented to them. The allowed scale to rate the quality is defined in the range from 1 (very poor quality) to 5 (brilliant quality).

Table 4.1 summarizes the different kinds of QoS related measurements, typical outcomes and some examples.

Table 4.1: QoS related measurements, typical outcomes and examples

Category	Relevant measurement types	Examples
Binary values	Service accessibility, service availability Service retainability, service continuity Error ratios, error probabilities	Service accessibility telephony, service non-availability SMS Call completion rate, call drop rate Call set-up error rate
Duration values	Duration of a session or call Service access delay Round trip time, end-to-end delay Blocking times, system downtimes	Mean call duration Service access delay WAP ICMP Ping roundtrip time Blocking time telephony, SGSN downtime
Throughput values	Throughput	Mean data rate GPRS Peak data rate UMTS
Content quality values	Audio-visual quality	MOS scores out of subjective testing

5 Distributions and moments

5.1 Introduction

The objective of data analyses is to draw conclusions about the state of a process based on a given data set, which may or may not be a sample of the population of interest. If distributions are assumed, these specify the shape of the data mass up to parameters associated with each family of distributions specifying properties like the mean of the data mass. Location or dispersion shifts of the process will in general result in different parameter estimates specifying the distribution. Therefore the information available from the data is compressed into one or few sufficient statistics specifying the underlying distribution.

Many statistical applications and computations rely in some sense on distributional assumptions, which are not always explicitly stated. Results of statistical measures are often only sensible if underlying assumptions are met and therefore only interpretable if users know about these assumptions.

This clause is organized as follows. Firstly, distributions, moments and quantiles are introduced in theory in clauses 5.2 to 5.4. This part of the present document is based on the idea of random variables having certain distributions. Random variables do not take single values but describe the underlying probability model of a random process. They are commonly denoted by:

$$X \sim \text{distribution (parameters)}$$

From the distributional assumptions, moments and quantiles of random variables are derived in theory.

Data is often viewed as being realizations of random variables. Therefore, data analysis mainly consists of fitting an appropriate distribution to the data and drawing conclusions based on this assumption. Clause 5.5 briefly summarizes the estimation of moments and quantiles.

Subsequently, a number of important distributions is introduced in clause 5.6, each of which is visualized graphically to give an idea of meaningful applications. Within this clause, testing distributions are also introduced as they are needed in clause 5.7 for the derivation of statistical tests.

5.2 Continuous and discrete distributions

The main difference between the data types described above can be explained in terms of continuous and discrete distributions. Data with binary values follow a discrete distribution, since the probability mass is distributed only over a fixed number of possible values. The same holds for quality measurements with evaluation results on a scale with a limited number of possible values (i.e. marks 1 to 6 or similar).

On the contrary, time-interval measurements as well as quality measurements based on appropriate quantitative variables may take an infinitely large number of possible values. In theory, since the number of possible outcomes equals infinity, the probability that a single value is exactly realized is zero. Probabilities greater than zero are only realized for intervals with positive width. In practice, each measurement tool will only allow a limited precision resulting in discrete measurements with a large number of possible outcomes. Nevertheless, data from measurement systems with reasonable precision are treated as being continuous.

Formal definitions for continuous and discrete distributions are based on probability density functions as described in the following clauses.

5.3 Definition of density function and distribution function

5.3.1 Probability Distribution Function (PDF)

Probability Density Functions (PDF) specify the probability mass either for single outcomes (discrete distributions) or for intervals (continuous distributions).

A PDF is defined as a function $f : \mathcal{R} \rightarrow [0, \infty)$ with properties:

- i) $f(x) \geq 0$ for all $x \in \mathcal{S}$.
- ii) $\int_{\mathcal{S}} f(x) dx = 1$ for continuous distributions or $\sum_{\mathcal{S}} f(x) = 1$ for discrete distributions.

In other words, firstly the values of the PDF are always non-negative, meaning that negative probabilities are neither assigned to values nor intervals, and secondly the summation or integration over the PDF always results in 1 (= 100 %), meaning that any data value will always be realized.

EXAMPLE 1: A PDF for binary data may be given by $f(x) = \begin{cases} 0,1 & : x = 1 \\ 0,9 & : x = 0 \end{cases}$, which implies that the probability for a faulty trial ($x=1$) is 10 %, while tests are completed successfully with probability 90 %.

EXAMPLE 2: For time-interval measurements PDFs may take any kind of shape, as an example a normal distribution with mean 10 (seconds) is assumed here. The PDF for this distribution is given by $f(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x-10)^2\right\}$. Other examples for continuous distributions are following later on.

EXAMPLE 3: If for instance categories for speech quality are defined as 1 = very poor up to 5 = brilliant, a PDF for the resulting data may be given by $f(x) = \begin{cases} 0,1 & : x \in \{1,2,3\} \\ 0,4 & : x=4 \\ 0,3 & : x=5 \end{cases}$.

Figure 5.1 summarizes all three assumed example PDFs for the different data types.

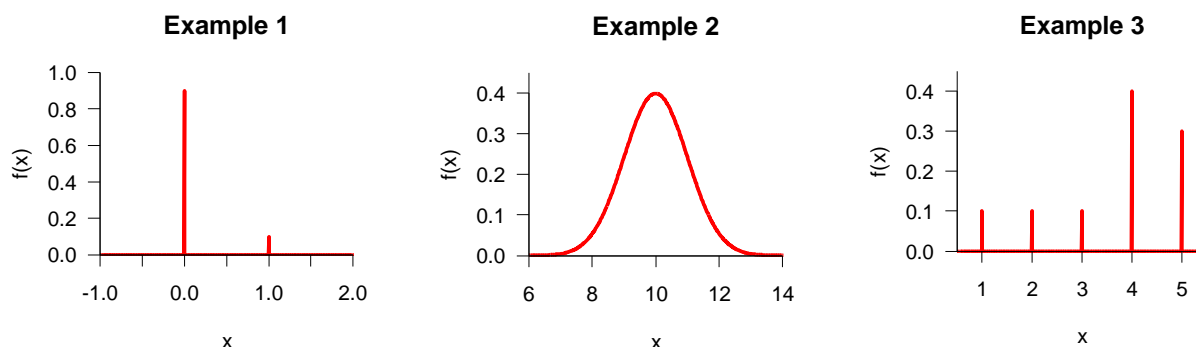


Figure 5.1: Probability Density Functions (PDFs) of examples 1 to 3

5.3.2 Cumulative Distribution Function (CDF)

A Cumulative Distribution (or Density) Function (CDF) is computed from the corresponding PDF as described before by summing (discrete) or integrating (continuous) over the density mass up to the current value.

A function $F: \mathbb{R} \rightarrow [0,1]$ with $F(x) = \sum_{\tilde{x} \leq x} f(\tilde{x})$ for discrete and $F(x) = \int_{-\infty}^x f(\tilde{x}) d\tilde{x}$ for continuous distributions is called CDF. This implies $F(x) \rightarrow 1$ for $x \rightarrow \infty$ and $F(x) \rightarrow 0$ for $x \rightarrow -\infty$.

In other words, the value of the CDF corresponds to the proportion of the distribution left of the value of interest. For the three examples from above, the CDFs are given in figure 5.2.

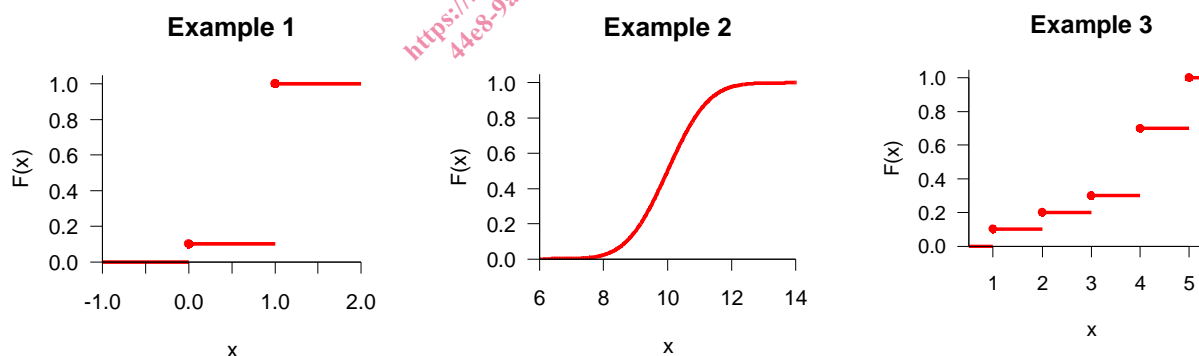


Figure 5.2: Cumulative Distribution Functions (CDFs) of examples 1 to 3

5.4 Moments and quantiles

Moments are main characteristics of distributions. The most important moments are:

- the **expected value** (first moment), specifying the location of the distribution;
- the **variance** (second central moment), specifying the dispersion around the expected value of the distribution; and

- the **skewness** (third central moment), specifying whether a distribution is symmetric or skewed.

These moments are defined as follows.

- a) The expected value (first moment, mean) of a random variable x with CDF $f(x)$ is defined as

$$E(x) = \int x \cdot f(x) dx \text{ for continuous distributions or } E(x) = \sum x \cdot f(x) \text{ for discrete distributions, respectively.}$$

- b) The variance (second central moment) of a random variable x with CDF $f(x)$ is defined as

$$Var(x) = \int (x - E(x))^2 \cdot f(x) dx \text{ for continuous distributions or } Var(x) = \sum (x - E(x))^2 \cdot f(x) \text{ for discrete distributions, respectively. The square root of the variance called standard deviation, denoted as } \sigma(x) \text{ is often more informative since it is defined on the original data scale.}$$

- c) The skewness (third central moment) of a random variable x with CDF $f(x)$ is defined as

$$\int (x - E(x))^3 \cdot f(x) dx \text{ for continuous distributions or } \sum (x - E(x))^3 \cdot f(x) \text{ for discrete distributions, respectively. A value of zero indicates a symmetric distribution.}$$

EXAMPLE 1: For the CDF from example 1 the moments are given by $E(x) = 0,1 \cdot 1 + 0,9 \cdot 0 = 0,1$, $Var(x) = 0,1 \cdot 0,9^2 + 0,9 \cdot 0,1^2 = 0,09$ resulting in a standard deviation $\sigma(x) = 0,3$. The skewness can be computed as $0,1 \cdot 0,9^3 + 0,9 \cdot (-0,1)^3 = 0,072$ indicating that the distribution is not symmetric.

EXAMPLE 2: The moments of the above normal distribution can be computed by partial integration and the fact that the PDF integrates to 1, or by utilizing the properties of normal distributions stating that the mean and standard deviation are the parameters μ and σ of the PDF

$$f(x | \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(x-10)^2\right\} \text{ and that normal distributions are always symmetric.}$$

This results in $E(x) = 10$, $Var(x) = \sigma^2 = 1$, which also equals the standard deviation and skewness = 0 for the above example.

EXAMPLE 3: For the CDF of example 3, moments are computed by $E(x) = 0,1 \cdot 1 + 0,1 \cdot 2 + 0,1 \cdot 3 + 0,4 \cdot 4 + 0,3 \cdot 5 = 3,7$, $Var(x) = 1,61$ and negative skewness of $-1,824$.

The moments are computable for all three example PDFs. Nevertheless, they are not always meaningful. In particular in the third example, the possible outcomes are "very poor" to "brilliant", which may be ordered and named 1 to 5 as has been done before, but the expected value of 3,7 does not have a strict meaning. The same applies for higher moments, since the values of the variable of interest are not quantitative, but ordered qualitative.

In case of non-symmetric distributed data, moments may not be appropriate for describing the distribution of interest. An alternative measure of location is given by the **median**, which can be viewed as the point cutting the distribution into halves, namely 50 % of the distribution mass are smaller and 50 % are larger than the median.

More generally, quantiles are defined for each possible percentage. The α -**quantile** cuts the distribution in a part of $\alpha \cdot 100$ % of the distribution smaller than this value and $(1-\alpha) \cdot 100$ % larger than this value. The median as a special case is also called 50 %-quantile.

A **formal definition of quantiles** is for instance given by Mood, Graybill, Boes (1974):

- "The α -quantile q_α with $\alpha \in (0,1]$ is defined as the smallest number q_α satisfying $F(q_\alpha) \leq \alpha$ (for $\alpha = 0$, the minimum value with positive probability or $-\infty$ is defined, respectively)".

Quantiles are easiest illustrated with the examples of CDFs given above, compare figure 5.3. For each CDF, the 5 %, 50 % and 75 %-quantiles are added to the corresponding plot.