10 Operations

10.1 Overview

This document specifies operations on SRF coordinates and, in the case of 3D object-spaces, on SRF spatial directions, vectors and orientations. All SRFs rely on their respective embedded frames in object-space to provide a reference orthonormal frame for such operations. Underlying some of these operations are the similarity transformations relating two ORMs (two SRFs with the same ORM is treated as a special case). Similarity transformations are treated first in 10.3. The general case of changing the coordinate of a position in one SRF to its corresponding coordinate in another SRF is specified in 10.4, followed by important special cases. The specification of a spatial direction, vector, or orientation in the context of an SRF is defined, and operations for changing these representations from one SRF to their corresponding representations in another SRF are specified in 10.5.

Euclidean distance in 2D and 3D object-space is specified in 10.6. Geodesic distance and azimuth on the surface of an oblate ellipsoid (or sphere) are specified in 10.7.

10.2 Symbols and terminology

An important category of spatial operations is changing the representation of spatial information in one SRF to the representation in a second SRF. For these SRF operations, the adjective "source" shall be used to refer to the first SRF, and the adjective "target" shall be used to refer to the second SRF.

The symbols in Table 10.1 are used throughout this clause.

Table 10.1 — Symbols

Symbol	Definition
CS_{S}	spatial coordinate system of $\mathrm{SRF}_{\mathrm{S}}$
$\mathrm{ORM}_{\mathrm{S}}$	object reference model of $\mathrm{SRF}_{\mathrm{S}}$
$\mathrm{ORM}_{\mathrm{R}}$	reference ORM for a given spatial object
$\mathrm{SRF}_{\mathrm{S}}$	source spatial reference frame
$\mathrm{SRF}_{\mathrm{T}}$	target spatial reference frame
$\boldsymbol{c}_{\mathrm{S}}$	coordinate of a position in $\mathrm{SRF}_{\mathrm{S}}$
$d_E()$	Euclidean distance
$d_G()$	geodesic distance
$\boldsymbol{E}$	normal embedding
E_{S}	extended region of $\mathrm{SRF}_{\mathrm{S}}$
$\boldsymbol{E}_{\mathrm{S}}$	embedded frame of $\mathrm{SRF}_{\mathrm{S}}$
$\boldsymbol{G}_{\mathrm{S}}$	spatial generating function of CS_{S}
$Dom(\boldsymbol{G}_{\mathrm{S}})$	domain of the generating function $\boldsymbol{G}_{\mathrm{S}}$
$Rng(\boldsymbol{G}_{\mathrm{S}})$	range of the generating function $\boldsymbol{G}_{\mathrm{S}}$
$\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$	similarity transformation from frame S to frame T
$\boldsymbol{I}$	identity matrix (or operator)
$\boldsymbol{L}$	localized orthonormal frame
$\boldsymbol{L}_{3\mathrm{D}}$	localization operator (3D)

Symbol	Definition
$\boldsymbol{M}_{\mathrm{T}\leftarrow\mathrm{S}}$	rotation matrix from frame *S* to frame T
$\boldsymbol{n}_{\mathrm{S}}$	direction vector in $\mathrm{SRF_S}$
$\boldsymbol{P}_{\mathrm{S}}$	mapping equations for $\mathrm{SRF_S}$
$\boldsymbol{p}$	position vector
$\boldsymbol{Q}_{\mathrm{S}}$	inverse mapping equations for $\mathrm{SRF_S}$
$\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t}$	localization parameters
$\boldsymbol{R}$	rotation operator
V_{S}	applicable region of $\mathrm{SRF_S}$
$\boldsymbol{v}_{\mathrm{S}}$	vector quantity in $\mathrm{SRF_S}$
$\boldsymbol{W}$	world 3x3 transformation matrix
$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathrm{S}}$	3D position vector components in $\mathrm{SRF_S}$
$\vec{\Delta}_{\mathrm{T}\leftarrow\mathrm{S}}$	displacement vector from the origin of frame T to the origin of frame *S*
$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}_{\mathrm{S}}$	origin displacement vector components in $\mathrm{SRF_S}$
$(\lambda_{\mathrm{S}}, \varphi_{\mathrm{S}}, h_{\mathrm{S}})$	geodetic coordinate tuple for a position in $\mathrm{SRF_S}$
$\sigma_{\mathrm{T}\leftarrow\mathrm{S}}$	scale factor from frame *S* to frame T
$\boldsymbol{\Omega}_{\mathrm{T}\leftarrow\mathrm{S}}$	change of basis operator from frame *S* to frame T

10.3 ORM operations

10.3.1 Overview

The similarity transformation (see 7.3.2) $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$ between two object reference models, source $\mathrm{ORM_S}$ and target $\mathrm{ORM_T}$ underlies the coordinate operations in 10.4. There are two cases, depending on whether $\mathrm{ORM_S}$ and $\mathrm{ORM_T}$ represent the same object, or represent two different objects.

The case where $\mathrm{ORM_S}$ and $\mathrm{ORM_T}$ represent the same object is addressed in 10.3.2. Although objects are often represented by only a single object reference model, some objects, such as the Earth, are represented by many different object reference models (see Annex E). Given a set of n object reference models for an object, there are $n(n-1)$ possible source and target ORM pairs. Instead of specifying all possible similarity transformations among these object reference models, this document reduces the requirement to specifying the reference transformation $\boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{S}}$ from each source ORM for the object, $\mathrm{ORM_S}$ to the designated reference ORM for the object, $\mathrm{ORM_R}$.

The more general case where $\mathrm{ORM_S}$ and $\mathrm{ORM_T}$ represent two different objects is addressed in 10.3.3. This includes subcases where one or both objects are represented by multiple object reference models, and where $\mathrm{ORM_S}$ and/or $\mathrm{ORM_T}$ are not the reference object reference models for their respective objects. It also includes subcases with different types of relationships between the two objects (see 8.4).

10.3.2 Relating different ORMs for the same object

If $\mathrm{ORM_S}$ and $\mathrm{ORM_T}$ are different object reference models that represent the same object, and therefore share the same reference ORM, $\mathrm{ORM_R}$, the similarity transformation $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$ is the composition of their reference transformations $\boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{S}}$ and $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{R}}$, the inverse of $\boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{T}}$ as shown in Figure 10.1. This is the common datum transformation operation.

$$\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}} = \boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{R}} \circ \boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{S}} \quad (10.1)$$

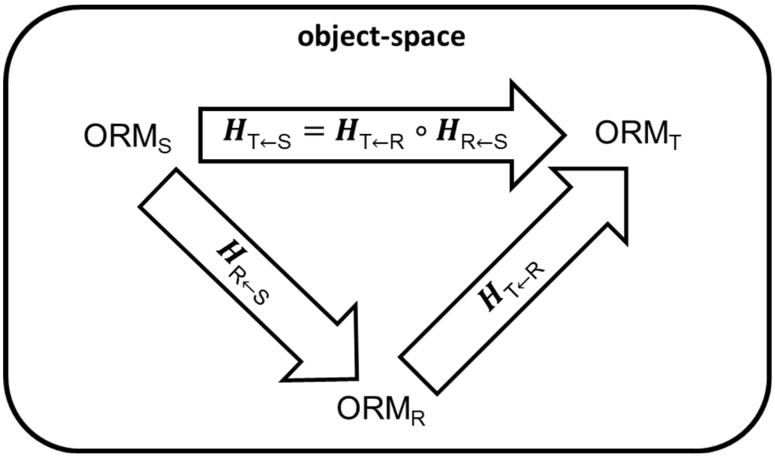

Figure 10.1 — Composed transformations for a single object

If $\mathrm{ORM_S}$ is the reference ORM for the object, $\boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{S}}$ reduces to the identity $\boldsymbol{I}$. Similarly, if $\mathrm{ORM_T}$ is the reference ORM for the object, $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{R}}$ reduces to the identity $\boldsymbol{I}$.

If $\mathrm{ORM_S}$ and $\mathrm{ORM_T}$ are identical, the similarity transformation $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$ reduces to the identity $\boldsymbol{I}$ (see 10.4.3 and 10.4.4). This subcase includes the relationship between a regional SRF and another SRF used as a reference (see 8.4.2).

If $\mathrm{ORM_S}$ is an object-fixed ORM, its reference transformation $\boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{S}}$ is a type of similarity transformation. Any 3D or 2D similarity transformation may be represented with the STT ROTATE_SCALE_TRANSLATE in the 3D case or STT ROTATE_SCALE_TRANSLATE_2D in the 2D case. Thus, using the notation of the STT formulation, $\boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{S}}$ may be represented as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathrm{R}} = \boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{S}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathrm{S}}\right) \equiv \vec{\Delta}_{\mathrm{R}\leftarrow\mathrm{S}} + \sigma_{\mathrm{R}\leftarrow\mathrm{S}}\boldsymbol{M}_{\mathrm{R}\leftarrow\mathrm{S}}\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathrm{S}} \quad (10.2)$$

NOTE For the Earth, the processes by which object reference models are established are based on physical measurements. These measurements are subject to error, and therefore introduce various types of relative distortions between object reference models. The scale factor $\sigma_{\mathrm{R}\leftarrow\mathrm{S}}$ in Equation 10.2 should equal 1,0 since each ORM is for the same object-space. However, values very close to 1,0 are allowed to account for small distortions (see 7.3.2). The reference transformation $\boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{T}}$ from $\mathrm{ORM_T}$ to the reference $\mathrm{ORM_R}$ is also a similarity transformation.

The similarity transformation $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{R}}$ is:

$$\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{R}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathrm{R}}\right) = \boldsymbol{H}^{-1}_{\mathrm{R}\leftarrow\mathrm{T}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathrm{R}}\right) = \left({}^{1}/_{\sigma_{\mathrm{R}\leftarrow\mathrm{T}}}\right)\boldsymbol{M}^{-1}_{\mathrm{R}\leftarrow\mathrm{T}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathrm{R}} - \vec{\Delta}_{\mathrm{R}\leftarrow\mathrm{T}}\right)$$

$$= \vec{\Delta}_{\mathrm{T}\leftarrow\mathrm{R}} + \left({}^{1}/_{\sigma_{\mathrm{R}\leftarrow\mathrm{T}}}\right)\boldsymbol{M}^{-1}_{\mathrm{R}\leftarrow\mathrm{T}}\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathrm{R}}$$

Because the matrix $\boldsymbol{M}_{\mathrm{R}\leftarrow\mathrm{T}}$ is a rotation matrix, its transpose $\boldsymbol{M}^{\top}_{\mathrm{R}\leftarrow\mathrm{T}}$ is also its inverse $\boldsymbol{M}^{-1}_{\mathrm{R}\leftarrow\mathrm{T}}$. The inverse of $\boldsymbol{M}_{\mathrm{R}\leftarrow\mathrm{T}}$ is also the matrix $\boldsymbol{M}_{\mathrm{T}\leftarrow\mathrm{R}}$ corresponding to the reverse rotations of $\mathrm{ORM_T}$ with respect to $\mathrm{ORM_R}$. In particular:

$$\boldsymbol{M}_{\mathrm{T}\leftarrow\mathrm{R}} = \boldsymbol{M}^{-1}_{\mathrm{R}\leftarrow\mathrm{T}} = \boldsymbol{M}^{\top}_{\mathrm{R}\leftarrow\mathrm{T}}$$

and

$$\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{R}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathrm{R}}\right) = \vec{\Delta}_{\mathrm{T}\leftarrow\mathrm{R}} + \left({}^{1}/_{\sigma_{\mathrm{R}\leftarrow\mathrm{T}}}\right)\boldsymbol{M}_{\mathrm{T}\leftarrow\mathrm{R}}\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathrm{R}}.$$

The composite operation $\boldsymbol{H}_{\text{T}\leftarrow\text{S}} = \boldsymbol{H}_{\text{T}\leftarrow\text{R}} \circ \boldsymbol{H}_{\text{R}\leftarrow\text{S}}$ reduces to:

$$\boldsymbol{H}_{\text{T}\leftarrow\text{S}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\text{S}}\right) = \boldsymbol{H}_{\text{T}\leftarrow\text{R}} \circ \boldsymbol{H}_{\text{R}\leftarrow\text{S}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\text{S}}\right) = \vec{\Delta}_{\text{T}\leftarrow\text{S}} + \left({}^{\sigma_{\text{R}\leftarrow\text{S}}}/_{\sigma_{\text{R}\leftarrow\text{T}}}\right)\boldsymbol{M}_{\text{T}\leftarrow\text{S}}\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\text{S}} \tag{10.3}$$

where:

$\boldsymbol{M}_{\text{T}\leftarrow\text{S}} = \boldsymbol{M}_{\text{T}\leftarrow\text{R}}\,\boldsymbol{M}_{\text{R}\leftarrow\text{S}}$, and $\vec{\Delta}_{\text{T}\leftarrow\text{S}} = \vec{\Delta}_{\text{T}\leftarrow\text{R}} + \left({}^{1}/_{\sigma_{\text{R}\leftarrow\text{T}}}\right)\boldsymbol{M}_{\text{T}\leftarrow\text{R}}\,\vec{\Delta}_{\text{R}\leftarrow\text{S}}$

If the rotations $\boldsymbol{M}_{\text{R}\leftarrow\text{S}}$ and $\boldsymbol{M}_{\text{R}\leftarrow\text{T}}$ are equal, then $\boldsymbol{M}_{\text{T}\leftarrow\text{S}}$ is the identity matrix, and if $\sigma_{\text{R}\leftarrow\text{S}} = \sigma_{\text{R}\leftarrow\text{T}}$, $\boldsymbol{H}_{\text{T}\leftarrow\text{S}}$ simplifies to a translation of the origin:

$$\boldsymbol{H}_{\text{T}\leftarrow\text{S}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\text{S}}\right) = \vec{\Delta}_{\text{T}\leftarrow\text{S}} + \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\text{S}}.$$

Equation 10.1 and Figure 10.1 also apply to the 2D case.

If the source ORMs is a time-dependent ORM for a spatial object, ORMs(t) shall denote the source ORMs at time t, and $\boldsymbol{H}_{\text{R}\leftarrow\text{S}}(t)$ shall denote the similarity transformation from ORMs(t) to the object-fixed reference ORMR. For a fixed value of time t_0, Equation 10.1 and Figure 10.1 generalize to $\boldsymbol{H}_{\text{T}\leftarrow\text{S}}(t_0) = \boldsymbol{H}_{\text{T}\leftarrow\text{R}} \circ \boldsymbol{H}_{\text{R}\leftarrow\text{S}}(t_0)$. The generalization to a time-dependent target ORMT(t) is $\boldsymbol{H}_{\text{T}\leftarrow\text{S}}(t_0) = \boldsymbol{H}_{\text{T}\leftarrow\text{R}}(t_0) \circ \boldsymbol{H}_{\text{R}\leftarrow\text{S}}$. The generalization when both ORMs are time-dependent at time t_0 is $\boldsymbol{H}_{\text{T}\leftarrow\text{S}}(t_0) = \boldsymbol{H}_{\text{T}\leftarrow\text{R}}(t_0) \circ \boldsymbol{H}_{\text{R}\leftarrow\text{S}}(t_0)$.

EXAMPLE ORMs(t) is the ORM EARTH_INERTIAL_J2000r0 at time t. ORMR is the Earth reference ORM WGS_1984. Because ORMs(t) and ORMR share the same embedding origin, the $\boldsymbol{H}_{\text{R}\leftarrow\text{S}}(t)$ transformation is a (rotation) matrix multiplication operation (without translation). The matrix coefficients for selected values of t account for polar motion, Earth rotation, nutation, and precession. Predicted values for these coefficients are computed and updated weekly by the International Earth Rotation and Reference Systems Service (IERS) [IERS36]. See 7.5 for other examples of dynamic ORM reference transformations.

10.3.3 Relating ORMs for different objects

If ORMs and ORMT are different object reference models that represent two different objects, a source object ***S*** and a target object ***T***, the similarity transformation $\boldsymbol{H}_{\text{T}\leftarrow\text{S}}$ is the composition of the reference transformation for ORMs, $\boldsymbol{H}_{\text{R}_\text{S}\leftarrow\text{S}}$, the similarity transformation between the reference object reference models of the two objects, $\boldsymbol{H}_{\text{R}_\text{T}\leftarrow\text{R}_\text{S}}$, and the inverse reference transformation for ORMT, $\boldsymbol{H}_{\text{T}\leftarrow\text{R}_\text{T}}$, as shown in Figure 10.2.

$$\boldsymbol{H}_{\text{T}\leftarrow\text{S}} = \boldsymbol{H}_{\text{T}\leftarrow\text{R}_\text{T}} \circ \boldsymbol{H}_{\text{R}_\text{T}\leftarrow\text{R}_\text{S}} \circ \boldsymbol{H}_{\text{R}_\text{S}\leftarrow\text{S}} \tag{10.4}$$

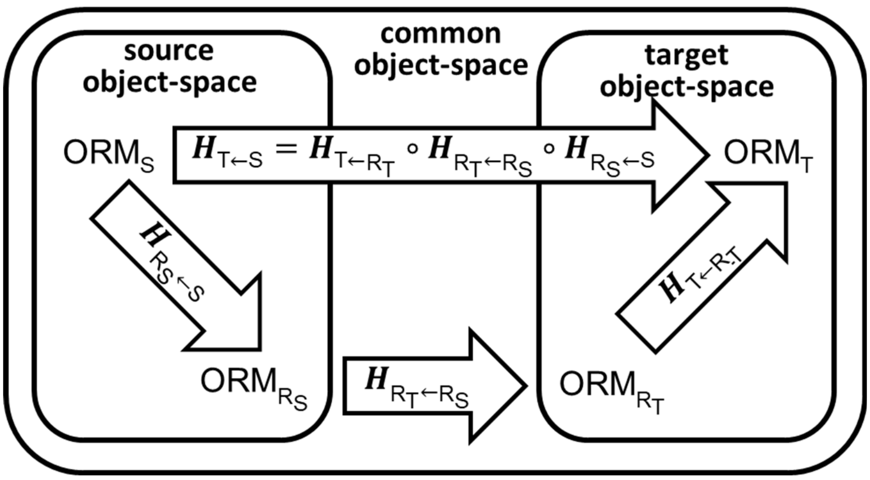

Figure 10.2 — Composed transformations for two different objects