5 Coordinate systems

5.1 Overview

Coordinate systems provide the mathematical underpinnings for spatial operations in multidimensional space. A coordinate system assigns coordinates to points in space and/or time. This document defines coordinate systems within the scope of three conceptual spaces: coordinate-space, position-space, and object-space. *Coordinate-spaces* specify the sets of coordinate n-tuples that form the domains of coordinate systems. *Position-spaces* are abstract Euclidean vector spaces that provide the mathematical and geometric foundation needed to define spatial operations. *Object-spaces* are Euclidean vector spaces associated with specific spatial objects of interest, such as the Earth, a building, or a vehicle. Coordinate-spaces, position-spaces, and object-spaces, and the relationships among them, are normatively defined in 5.2.

An *abstract coordinate system* specifies a function, termed a *generating function*, which assigns unique n-tuples in a domain in coordinate-space to points in an m-dimensional position-space $(1 \leq n \leq m \leq 3)$. Abstract coordinate systems are normatively defined, and many types of abstract coordinate systems are specified, in 5.3.

A *spatial coordinate system* extends the assignment of unique coordinate n--tuples from points in a position-space to points in an object-space. The assignment function combines an abstract coordinate system generating function with a *normal embedding* that maps the orthonormal frame within position-space to a corresponding orthonormal frame within object-space. Spatial coordinate systems are normatively defined in 5.4. The relationships among coordinate-space, position-space, abstract coordinate systems, object-space, and spatial coordinate systems are shown in Figure 5.23.

The ability of a spatial coordinate system to assign a unique coordinate to a point in an object-space assumes that the position of the point in object-space is static. In a dynamic system, that assumption may not hold unless the spatial coordinate system is associated with a particular moment in time. *Temporal coordinate systems* provide a standard way of associating time with a spatial coordinate system. Temporal coordinate systems are normatively defined in 5.5.

5.2 Coordinate-space, position-space, and object-space

5.2.1 Coordinate-space

A *coordinate*[2] is an ordered n-tuple $(1 \leq n \leq 3)$. A *coordinate-component*[3] is an individual element of a coordinate n-tuple. The k^{th} *coordinate-component* $(1 \leq k \leq 3)$ is the k^{th} component of a coordinate n-tuple. A coordinate system may optionally specify coordinate-component names and symbols in a specified order. In 3D coordinate systems, the 3rd coordinate-component may be identified as the *vertical coordinate-component*.

A *coordinate-space* specifies a set of coordinate n-tuples that forms the domain of a coordinate system. Such coordinate n-tuples include Cartesian (x, y, z), polar (ρ, θ), cylindrical (ρ, θ, h), and geodetic (λ, φ, h). A coordinate-space may include constraints on coordinate n-tuple components in such domains.

[2] The ISO 19111 term for this concept is “coordinate tuple”.

[3] The ISO 19111 term for this concept is “coordinate”.

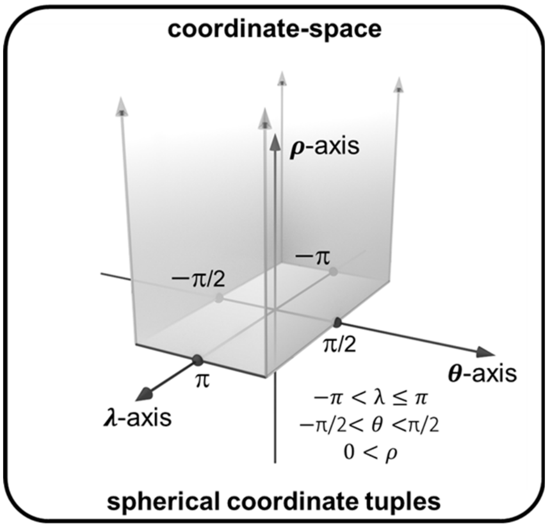

Figure 5.1 — A coordinate-space (including the domain for spherical coordinate n-tuples)

Figure 5.1 illustrates the structure of a coordinate-space for 3D spherical coordinate tuples of the form (λ, θ, ρ). The coordinate-components of these tuples are:

λ: longitude in radians, such that $-\pi < \lambda \leq \pi$,

θ: spherical latitude in radians, such that $-{}^{\pi}/_{2} < \theta < {}^{\pi}/_{2}$, and

ρ: radius in metres, such that $0 < \rho$.

This coordinate-space defines the domain for a spherical coordinate system (see 5.3.8.4) as a subset (highlighted in grey) of the coordinate-space.

5.2.2 Orthonormal frames

An *orthonormal frame* within a Euclidean vector space, in 2 or 3 dimensions, consists of an origin $\mathbf{0}$ and an ordered set of mutually perpendicular unit basis vectors $\boldsymbol{e_1}, \boldsymbol{e_2}$, and, in the 3D case, $\boldsymbol{e_3}$. Orthonormal frames provide uniform references for measuring distances and angles in both position-space and object-space and are the foundation upon which coordinate systems are constructed. The 3D case is depicted in Figure 5.2.

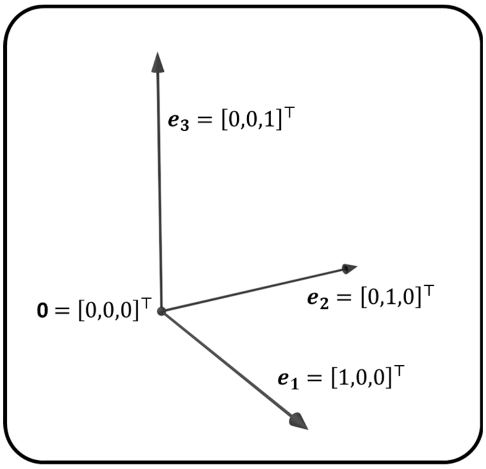

Figure 5.2 — A right-handed orthonormal frame

A 3D orthonormal frame is termed right-handed if the vertices of the triangle formed by its basis unit vectors are in clockwise order when viewed from the origin, as defined in ISO 80000-2, and shown in Figure 5.3. In this document, all 3D orthonormal frames shall be right-handed.

5.2.3 Position-space

Position-space of dimension m, $(1 \le n \le m \le 3)$, is the Euclidean vector space $\mathbb{R}^m$ as defined in A.2. Mathematical concepts of $\mathbb{R}^m$ as a vector space, the point-set topology of $\mathbb{R}^m$, the theory of real-valued functions on $\mathbb{R}^m$, and algebraic and analytic geometry, including the concepts of point, line, and plane, are all assumed and hold.

Position-space serves as a mathematical abstraction of object-spaces so that the methods of linear algebra and multivariate calculus can be applied to spatial concepts, including abstract coordinate systems and the computational aspects of spatial operations. The purpose of position-space is to provide flexibility in applying different types of coordinate systems to object-spaces for many different types of spatial objects of interest.

The *position* of a point is the displacement of that point with respect to a designated reference point, called the origin. Each point in Euclidean vector space is associated with the position vector that extends from the origin to that point with length equal to the Euclidean distance between the origin and that point. Thus, points in Euclidean space and position vectors with respect to the origin are equivalent concepts. The position of an object is typically expressed in terms of the position of a representative point within the object.

A *direction* in a Euclidean vector space is represented by a unit vector. *A vector quantity*, expressing a physical measurement such as velocity or acceleration (at a given instant in time), is represented by a direction vector combined with a magnitude. Velocity is a vector quantity that expresses the rate of change of position. Acceleration is a vector quantity that expresses the rate of change of velocity.

The orthonormal frame that is inherent to position-space forms a canonical Cartesian basis. This Cartesian basis allows positions, directions, vector quantities, and distance measurements in position-space to be quantified.

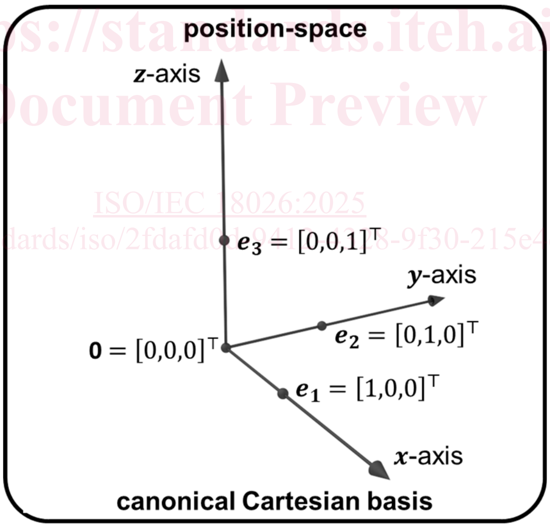

Figure 5.3 — 3D position-space, its orthonormal frame, and its canonical Cartesian basis

Figure 5.3 illustrates 3D position-space, showing its origin, its inherent orthonormal frame, and its canonical Cartesian basis. The axes of the Cartesian basis are aligned with the basis vectors of the orthonormal frame.

Position vectors in two and three dimensions are denoted as $[x, y]^\top \equiv \begin{bmatrix} x \\ y \end{bmatrix}$ and $[x, y, z]^\top \equiv \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, respectively (see A.2). These components, unless otherwise indicated, are specified with respect to the canonical Cartesian basis and origin.

The canonical origin for $\mathbb{R}^2$ is the zero vector $\mathbf{0} = [0,0]^\top$. The canonical Cartesian basis vectors for $\mathbb{R}^2$ are $\boldsymbol{e}_1 = [1,0]^\top, \boldsymbol{e}_2 = [0,1]^\top$.

The canonical origin for $\mathbb{R}^3$ is the zero vector $\mathbf{0} = [0,0,0]^\top$. The canonical Cartesian basis vectors for $\mathbb{R}^3$ are $\boldsymbol{e}_1 = [1,0,0]^\top$, $\boldsymbol{e}_2 = [0,1,0]^\top$, $\boldsymbol{e}_3 = [0,0,1]^\top$.

5.2.4 Object-space

Object-space is the Euclidean vector space (a universe[4]) that is fixed to a designated spatial object of interest. Object-space provides the application domain context for spatial concepts including positions, directions, vector quantities, and orientations.

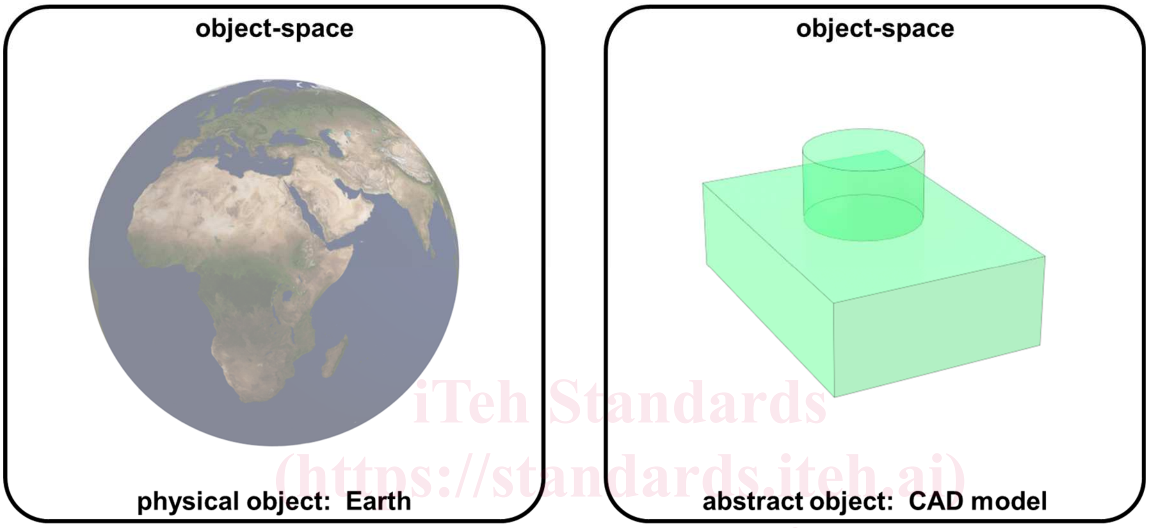

Figure 5.4 — Object-spaces for the Earth and for a CAD model

The spatial objects of concern in this document include physical and abstract objects, as illustrated in Figure 5.4. *Physical objects* are real-world objects, such as Earth or a building. The length of one metre has intrinsic meaning in the object-space of a physical object. *Abstract objects* are conceptual objects including engineering, mathematical, and virtual models. A length of one metre does not have intrinsic meaning in the object-spaces of abstract objects. Thus, to relate abstract object-spaces to other (physical or abstract) object-spaces, each abstract object-space is required to have a designated length scale.

At any given instance in time, the position of a point in object-space is fixed with respect to the spatial object of interest. This is done either by a time-invariant constant or a time-dependent function. If points and the spatial object of interest have a time-dependent relationship, the positions of the points shall be qualified by a time value. Thus, at a specified time, the points and the spatial object of interest have a fixed spatial relationship.

EXAMPLE 1 The Sun and the Earth are both physical objects. In the object-space of the Sun, the Sun is the spatial object of interest and is fixed and the Earth moves according to a time dependent function. In the object-space of the Earth, the Earth is the spatial object of interest and is fixed and the Sun moves according to a different time dependent function.

EXAMPLE 2 At any given time the International Space Station (ISS) has a unique and unambiguous position in the object-space of the Earth.

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[4] The set of all continuations of a spatial object is termed the universe of the object. In physics, this is termed “the space of the object”. [EINS]

EXAMPLE 3 At any given time each component of the ISS has a fixed position in the object-space of the ISS.

EXAMPLE 4 A solar collector component of the ISS was manufactured in compliance with an engineering model. The engineering model was designed in the object-space of an abstract CAD/CAM model. The physical solar collector was constructed in its own physical object-space.

An object-space is a Euclidian space. In general, however, an object-space is not a vector space. Once a point in object-space is designated as an origin point, it becomes a vector space with respect to that origin and all points in the object-space are vectors, each with length and direction given as the distance and direction of the point from the origin.

5.2.5 Normal embeddings

A *normal embedding* is a distance-preserving function mapping vectors in position-space to points in an object-space of the same dimension. A function $\boldsymbol{E}$ from position-space to object-space is *distance-preserving* if for any two positions $\boldsymbol{p}$ and $\boldsymbol{q}$ in position-space, the Euclidean distance $d(\boldsymbol{p},\boldsymbol{q})$ is equal to the measured distance in object-space from $\boldsymbol{E}(\boldsymbol{p})$ to $\boldsymbol{E}(\boldsymbol{q})$ in metres. The distance-preserving property implies that a normal embedding is one-to-one and continuous. Normal embeddings also preserve angles and areas.

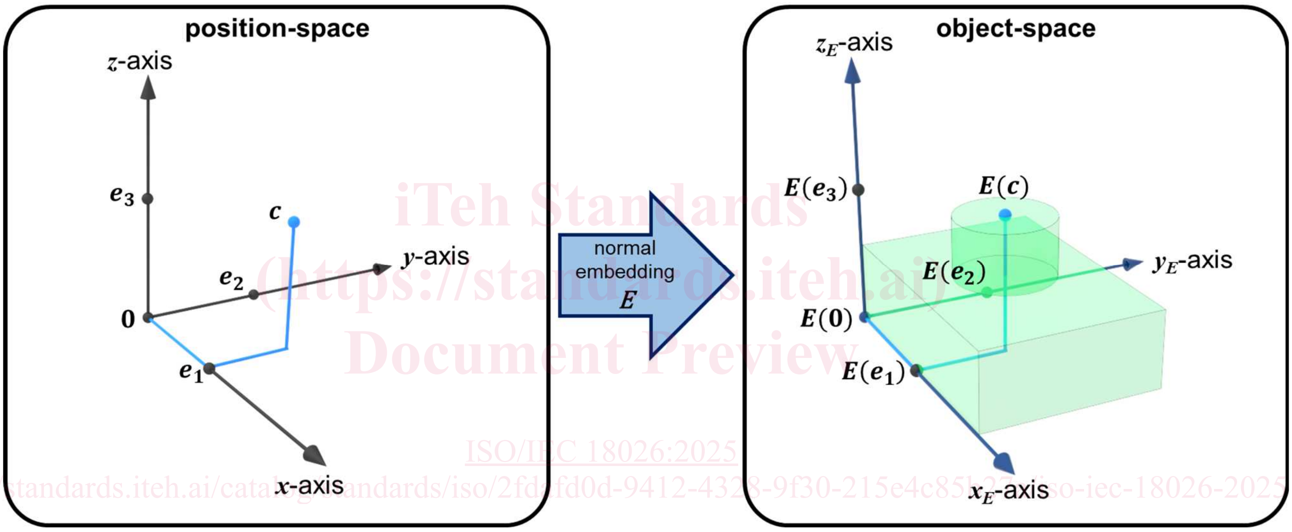

Figure 5.5 — A normal embedding that maps position-space to an object-space

Position-space together with a normal embedding provides a specific algebraic model of an object-space by determining an orthonormal frame within the object space. This frame is termed the *embedded frame* and is determined as follows. In the 3-dimentional case, as shown in Figure 5.5, the position-space orthonormal frame is formed by the origin $\boldsymbol{0}$ and unit basis vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$. The normal embedding $\boldsymbol{E}$ forms an orthonormal frame within object-space with origin $\boldsymbol{E}(\boldsymbol{0})$ and basis vectors $\boldsymbol{E}(\boldsymbol{e}_1)$, $\boldsymbol{E}(\boldsymbol{e}_2)$, and $\boldsymbol{E}(\boldsymbol{e}_3)$. Since $\boldsymbol{E}$ is distance preserving, these vectors are orthogonal unit vectors, thus an embedded frame is an orthonormal frame and $\boldsymbol{E}$ is then an isomorphism between position-space and object-space. A normal embedding of a 3D position-space is *right-handed* if this frame is a right-handed frame. Normal embeddings for 2-dimensional object-space form orthonormal frames in a similar way.

The point $\boldsymbol{E}(\boldsymbol{0})$ is termed the *origin of the normal embedding* $\boldsymbol{E}$. The point $\boldsymbol{E}(\boldsymbol{e}_1)$ is the *x_E-axis unit point* of the normal embedding $\boldsymbol{E}$. Depending on the dimension of position-space, $\boldsymbol{E}(\boldsymbol{e}_2)$ is the *y_E-axis unit point* and $\boldsymbol{E}(\boldsymbol{e}_3)$ is the *z_E-axis unit point*. Normal embeddings are used to relate abstract coordinate systems for position-space to spatial coordinate systems for an object-space (see 5.4).

There are infinitely many normal embeddings of an n-dimensional position-space for a given object-space, depending on placement of the origin and direction of the axes.

There are infinitely many ways to select the origin of the embedding in the object-space. The origin can be located at any point within the spatial object of interest, at any point on its surface, or at any point nearby in space. Common selections include the centre of mass of the object, its geometric centre, or a corner of the object (assuming it has corners) or its bounding volume such that the object is completely within the first octant.

Given a selected origin, there are infinitely many ways to orient the axes. If the object is a celestial body, the axes could be aligned with its rotational axis, its magnetic field axis, or the direction of the closest star (such as the Sun). If the object is a vehicle, the axes could be aligned based on its direction of forward motion or other common reference orientations. If the object is located on, or near, the surface of the Earth, common selections include east-north-up (ENU) and north-east-down (NED).

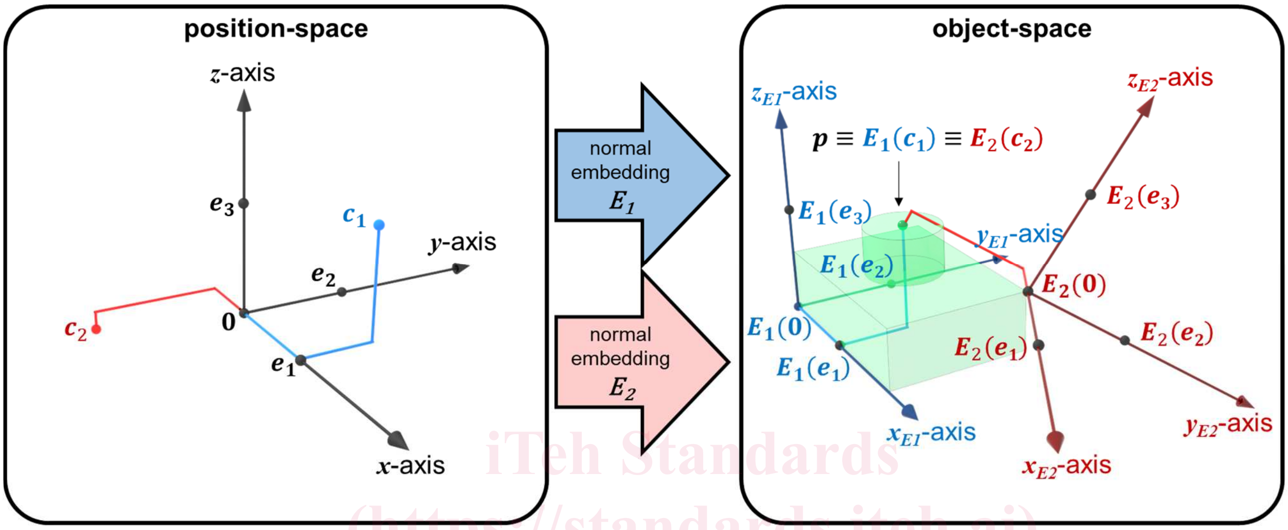

Figure 5.6 — Two distinct normal embeddings that map position-space to an object-space

Figure 5.6 illustrates two distinct normal embeddings for a given object-space, each determining a different embedded frame. Each embedding assigns the origin $(\boldsymbol{0})$ to different points on the spatial object of interest $(\boldsymbol{E_1}(\boldsymbol{0})$ and $\boldsymbol{E}_2(\boldsymbol{0})$, respectively), and assigning the basis vectors $(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3})$ to different directions $(\boldsymbol{E_1}(\boldsymbol{e1}), \boldsymbol{E_1}(\boldsymbol{e2}), \boldsymbol{E_1}(\boldsymbol{e3})$ and $\boldsymbol{E}_2(\boldsymbol{e1}), \boldsymbol{E}_2(\boldsymbol{e2}), \boldsymbol{E}_2(\boldsymbol{e3})$, respectively)) relative to the object, providing two distinct algebraic models of that object-space. In the figure, the two embedded frames are depicted in two different colours. In the object space, $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ refer to the same point $\boldsymbol{p} \equiv \boldsymbol{E}_1(\boldsymbol{c}_1)) \equiv \boldsymbol{E}_2(\boldsymbol{c}_2)$ on the object, expressed in each of the two embedded frames.

A similarity transformation is used to express the relationship between one embedded frame with respect to a second embedded frame within the same object-space. A similarity transformation consists of a translation, a rotation, and/or a scaling operation. If $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are two normal embeddings, there exists a similarity transformation $\boldsymbol{H_{E2 \leftarrow E1}}$ such that $\boldsymbol{E}_2$ is the composition of $\boldsymbol{E}_1$ with $\boldsymbol{H_{E2 \leftarrow E1}}$, i.e., $\boldsymbol{E}_2 = \boldsymbol{H_{E2 \leftarrow E1}} \circ \boldsymbol{E}_1$. This is depicted in Figure 5.6, where a point $\boldsymbol{p}$ in object-space will have vector coordinates $[x_1, y_1, z_1]_{\boldsymbol{E1}}$ and $[x_2, y_2, z_2]_{\boldsymbol{E2}}$ in the $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ embedded frames respectively. The similarity transformation $\boldsymbol{H_{E2 \leftarrow E1}}$, operating on object-space, that will translate $\boldsymbol{E}_1(\boldsymbol{0})$ to $\boldsymbol{E}_2(\boldsymbol{0})$ and align the $\boldsymbol{E}_1$ basis axes with $\boldsymbol{E}_2$ basis axes will also perform a change of basis operation: $[x_2, y_2, z_2]_{\boldsymbol{E2}} = H([x_1, y_1, z_1]_{\boldsymbol{E1}})$. Thus, similarity transformations can be used in the transformation of coordinates between orthonormal frames. Similarity transformations are addressed in greater detail in 7.3.2.

The method of specifying a normal embedding varies across disciplines and application domains. In some application domains, a normal embedding is implicitly defined by the specification of the origin point and axis directions. In the case of geodesy, an origin point at the centre of the Earth cannot be directly specified. Instead, its location is implied by specifying other geometric entities from physical measurements. An object reference model (see 7.4) implicitly identifies a unique normal embedding in this manner. Other disciplines use a variety of techniques to either implicitly or explicitly define a normal embedding. This document encapsulates these techniques within the concepts of reference datum and object reference model.

5.3 Abstract coordinate systems

5.3.1 Overview

An abstract coordinate system assigns a unique coordinate n-tuple to each point in a range of position vectors in an m-dimensional Euclidean vector space $(1 \leq n \leq m \leq 3)$ termed position-space, which has a canonical basis that defines an orthonormal frame. The assignment function is termed the generating function of the abstract coordinate system. The range may encompass the entire vector space or a proper sub-set, such as a surface or a curve.

Abstract coordinate systems are formally defined in 5.3.2. Abstract coordinate systems are characterized by type (5.3.3) and properties (5.3.5). In addition, abstract coordinate systems for 3D position-space generate coordinate-component surfaces (5.3.4). Localization operators modify abstract coordinate systems by shifting the vector space origin and changing axis directions (5.3.6). Map projections and augmented map projections are treated as a special case of abstract coordinate systems and have additional classifications and properties, as well as several functions unique to map projections (5.3.7). The elements for the specification of an abstract coordinate system, along with standardized abstract coordinate systems, are specified in 5.3.8. In this document the term “coordinate system (CS)”, if not otherwise qualified, is defined to mean “abstract CS.”

This document takes a functional approach to the construction of coordinate systems. Annex A provides a concise summary of mathematical concepts and specifies the notational conventions used in this document. In particular, Annex A defines the terms interior, one-to-one, smooth, smooth surface, smooth curve, orientation-preserving, and connected. Additionally, a newly introduced concept, replete, will be used. A set D is replete if all points in D belong to the closure of the interior of D (see Annex A). A replete set is a generalization of an open set that allows the inclusion of boundary points. Boundary points are important in the definitions of certain coordinate systems.

5.3.2 Definition

An *abstract coordinate system* (CS) assigns a unique coordinate to each point in a subset of position-space (5.2.3). An abstract Coordinate System shall be comprised of:

a) a CS domain in n-dimensional coordinate-space, $(1 \leq n \leq 3)$,

b) a generating function, and

c) a CS range in m-dimensional position-space, $(n \leq m \leq 3)$,

where:

a) The *CS domain* shall be a connected replete domain in n-dimensional coordinate-space, the space of n-tuples. The elements of the CS domain are coordinates.

b) The *generating function* assigns each coordinate to a point in position-space. It shall be a one-to-one, smooth function (see A.4) from the CS domain onto the generating function range.

c) The generating function range shall be termed the *CS range*. When $n = 2$ and $m = 3$, the CS range shall be a subset of a smooth surface[5]. When $n = 1$ and $m = 2$ or $m = 3$, the CS range shall be a subset of an implicitly specified smooth curve[6]. The elements of the CS range are positions.

[5] The generating function properties and the implicit function theorem together imply that for each point in the interior of the CS domain, there is an open neighbourhood of the point whose image under the generating function lies in a smooth surface. This requirement specifies that there exists one smooth surface for all of the points in the CS domain. This requirement is specified to exclude mathematically pathological cases.

[6] The generating function properties and the implicit function theorem together imply that for each point in the interior of the CS domain, there is an open neighbourhood of the point whose image under the generating function lies in a smooth curve. This requirement specifies that there exists one implicitly-defined smooth curve for all the points in the CS domain. This requirement is specified to exclude mathematically pathological cases.