Annex A
(normative)

Mathematical foundations

A.1 Overview

This annex identifies the concepts from mathematics used in this document and specifies the notation used for those concepts. A reader of this document is assumed to be familiar with mathematics including set theory, linear algebra, and the calculus of several real variables as presented in reference works such as the *Encyclopedic Dictionary of Mathematics* [EDM].

A.2 $\mathbb{R}^{\boldsymbol{n}}$ as a real vector space

An ordered set of n real numbers $\boldsymbol{a}$ where n is a natural number is termed an n*-tuple of real numbers* and shall be denoted by $\boldsymbol{a} = [a_1, a_2, a_3, \cdots, a_n]$ The set of all n-tuples of real numbers is denoted by $\mathbb{R}^n$. $\mathbb{R}^n$ is an n-dimensional vector space.

The *canonical basis* for $\mathbb{R}^n$ is defined as:

$$\boldsymbol{e}_1 = [1,0,\cdots,0],\ \boldsymbol{e}_2 = [0,1,\cdots,0],\cdots,\ \boldsymbol{e}_n = [0,0,\cdots,1].$$

The elements of $\mathbb{R}^n$ may be termed *points* or *vectors*. The latter term is used in the context of directions or vector space operations.

The zero vector $[0, 0, \cdots, 0]$ is denoted by $\boldsymbol{0}$.

Definitions A.2(a) through A.2(j) apply to any vectors $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]$ and $\boldsymbol{y} = [y_1, y_2, \cdots, y_n]$ in $\mathbb{R}^n$:

a) The *inner product* or *dot-product* of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as:

$$\boldsymbol{x} \bullet \boldsymbol{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

b) Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are termed *orthogonal* if $\boldsymbol{x} \bullet \boldsymbol{y} = 0$.

c) If $n \geq 2$, two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are termed *perpendicular* if and only if they are orthogonal.

NOTE 1 If $n \geq 2$, $\boldsymbol{x} \bullet \boldsymbol{y} = \|\boldsymbol{x}\| \|\boldsymbol{y}\| \cos(\alpha)$ where α is the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$.

d) $\boldsymbol{x}$ is termed *orthogonal to a set* of vectors if $\boldsymbol{x}$ is orthogonal to each vector that is a member of the set.

e) The *norm* of $\boldsymbol{x}$ is defined as

$$\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x} \bullet \boldsymbol{x}}.$$

NOTE 2 The norm of $\boldsymbol{x}$ represents the length of the vector $\boldsymbol{x}$. Only the zero vector $\boldsymbol{0}$ has norm zero.

f) $\boldsymbol{x}$ is termed a *unit vector* if $\|\boldsymbol{x}\| = 1$.

g) A set of two or more orthogonal unit vectors is termed an *orthonormal set of vectors*.

EXAMPLE The canonical basis is an example of an orthonormal set of vectors.

h) The *Euclidean metric* d is defined by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|.$$

i) The value of $d(\boldsymbol{x}, \boldsymbol{y})$ is termed the *Euclidean distance* between $\boldsymbol{x}$ and $\boldsymbol{y}$.

j) The *cross product* of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^3$ is defined as the vector:

$$\boldsymbol{x} \times \boldsymbol{y} = [x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1].$$

NOTE 3 The vector $\boldsymbol{x} \times \boldsymbol{y}$ is orthogonal to both $\boldsymbol{x}$ and $\boldsymbol{y}$, and

$$\|\boldsymbol{x} \times \mathbf{y}\| = \|\boldsymbol{x}\|\|\boldsymbol{y}\| \sin(\alpha),$$

where α is the angle between vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

A.3 The point set topology of $\mathbb{R}^n$

Given a point $\boldsymbol{p}$ in $\mathbb{R}^n$ and a real value $\varepsilon > 0$, the set $\{\boldsymbol{q}\ in\ \mathbb{R}^n | d(\boldsymbol{p}, \boldsymbol{q}) < \varepsilon\}$ is termed the ε-*neighbourhood* of $\boldsymbol{p}$.

Given a set $D \subset \mathbb{R}^n$ and a point $\boldsymbol{p}$**,** the following terms are defined:

a) $\boldsymbol{p}$ is an *interior point of* D if at least one ε-neighbourhood of $\boldsymbol{p}$ is a subset of D.

b) The *interior of a set* D is the set of all points that are interior points of D.

NOTE 1 The interior of a set may be empty.

c) D is *open* if each point of D is an interior point of D. Consequently, D is open if it is equal to its interior.

d) $\boldsymbol{p}$ is a *closure point of* D if every ε-neighbourhood of $\boldsymbol{p}$ has a non-empty intersection with D.

NOTE 2 Every member of D is a closure point of D.

e) The *closure of a set* D is the set of all points that are closure points of D.

f) D is a *closed set* if it is equal to the closure set of D.

g) A set D is *replete* if all points in D belong to the closure of the interior of D.

NOTE 3 Every open set is replete. The union of an open set with any or all its closure points forms a replete set. In particular, the closure of an open set is replete.

EXAMPLE 1 In $\mathbb{R}^2$ $\{(\boldsymbol{x}, \boldsymbol{y}) | -\pi < \boldsymbol{x} < \pi, -\pi/2 < \boldsymbol{y} < \pi/2\}$ is open and therefore replete.

EXAMPLE 2 $\{(\boldsymbol{x}, \boldsymbol{y}) | -\pi < \boldsymbol{x} \le \pi, -\pi/2 < \boldsymbol{y} < \pi/2\}$ is replete.

EXAMPLE 3 $\{(\boldsymbol{x}, \boldsymbol{y}) | -\pi < \boldsymbol{x} \le \pi, -\pi/2 \le \boldsymbol{y} \le \pi/2\}$ is closed and replete.

A.4 Smooth functions on $\mathbb{R}^n$

A real-valued function f defined on a replete domain in $\mathbb{R}^n$ is termed *smooth* if it is continuous and its first derivative exists and is continuous at each point in the interior of its domain.

The *gradient* of f is the vector of first order partial derivatives

$$\mathbf{grad}(f) = \left[\frac{\partial f}{\partial v_1}, \frac{\partial f}{\partial v_2}, \cdots, \frac{\partial f}{\partial v_n}\right].$$

Definitions A.4(a) through A.4(g) apply to any vector-valued function $\boldsymbol{F}$ defined on a replete domain D in $\mathbb{R}^n$ with range in $\mathbb{R}^m$.

a) The j^{th}-*component function* of a vector-valued function $\boldsymbol{F}$ is the real-valued function f_j defined by $f_j = \boldsymbol{e}_j \bullet \boldsymbol{F}$ where $\boldsymbol{e}_j$ is the j^{th} canonical basis vector, $j = 1,\ 2, \cdots, m$.

In this case:

$$\boldsymbol{F}(\boldsymbol{v}) = [f_1(\boldsymbol{v}), f_2(\boldsymbol{v}), f_3(\boldsymbol{v}), \ldots, f_m(\boldsymbol{v})] \text{ for } \boldsymbol{v} = [v_1, v_2, v_3, \ldots, v_n] \text{ in } D.$$

b) $\boldsymbol{F}$ is termed *smooth* if each component function f_j is smooth.

c) The *first derivative* of a smooth vector-valued function $\boldsymbol{F}$, denoted $\partial\boldsymbol{F}$, evaluated at a point in the domain is the $n \times m$ matrix of partial derivatives evaluated at the point:

$$\left[\frac{\partial f_j}{\partial v_i}\right] \; i = 1,\ 2,\cdots,n \text{ and } j = 1,2, \quad \cdots, m.$$

d) The *Jacobian matrix* of $\boldsymbol{F}$ at the point $\boldsymbol{v}$ is the matrix of the first derivative of $\boldsymbol{F}$.

NOTE 1 The rows of the Jacobian matrix are the gradients of the component functions of $\boldsymbol{F}$.

e) In the case $m = n$, the Jacobian matrix is square, and its determinant is termed the *Jacobian determinant.*

f) In the case $m = n$, $\boldsymbol{F}$ is termed *orientation preserving* if its Jacobian determinant is strictly positive for all points in D.

g) A vector-valued function $\boldsymbol{F}$ defined on $\mathbb{R}^n$ is *linear* if:

$$\boldsymbol{F}(a\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{F}(y) \text{ for all real scalars } a \text{ and vectors } \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ in } \mathbb{R}^n$$

NOTE 2 All linear functions are smooth.

A vector-valued function $\boldsymbol{E}$ defined on $\mathbb{R}^n$ is *affine* if $\boldsymbol{F}$, defined by $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{E}(\boldsymbol{x}) - \boldsymbol{E}(\boldsymbol{0})$, is a linear function. All affine functions on $\mathbb{R}^n$ are smooth.

A function may be alternatively termed an *operator* especially when attention is focused on how the function maps a set of points in its domain onto a corresponding set of points in its range.

EXAMPLE The localization operators (see 5.3.6.2).

A.5 Functional composition

If $\boldsymbol{F}$ and $\boldsymbol{G}$ are two vector valued functions and the range of $\boldsymbol{G}$ is contained in the domain of $\boldsymbol{F}$, then $\boldsymbol{F} \circ \boldsymbol{G}$, the *composition* of $\boldsymbol{F}$ with $\boldsymbol{G}$, is the function defined by $\boldsymbol{F} \circ \boldsymbol{G}(\boldsymbol{x}) \equiv \boldsymbol{F}\big(\boldsymbol{G}(\boldsymbol{x})\big)$. $\boldsymbol{F} \circ \boldsymbol{G}$ has the same domain as $\boldsymbol{G}$, and the range of $\boldsymbol{F} \circ \boldsymbol{G}$ is contained in the range of $\boldsymbol{F}$.

Functional composition also applies to scalar-valued functions f and g, If the range of g is contained in the domain of f, then $f \circ g(x)$, the composition of f with g, is the function defined by $f \circ g(x) \equiv f\big(g(x)\big)$.

A.6 Smooth surfaces in $\mathbb{R}^3$

A.6.1 Implicit definition

A *smooth surface* in $\mathbb{R}^3$ is *implicitly* specified by a real-valued smooth function f defined on $\mathbb{R}^3$ as the set S of all points (x, y, z) in $\mathbb{R}^3$ satisfying:

a) $f(x, y, z) = 0$ and

b) $\mathbf{grad}(f)(x, y, z) \neq \mathbf{0}$.

In this case, f is termed a *surface generating function* for the surface S.

EXAMPLE 1 If $\boldsymbol{n} \neq \mathbf{0}$ and $\boldsymbol{p}$ are vectors in $\mathbb{R}^3$ and $f(v) = \boldsymbol{n} \bullet (\boldsymbol{v} - \boldsymbol{p})$, then f is smooth and $\mathbf{grad}(f) = \boldsymbol{n} \neq \mathbf{0}$. The plane which is perpendicular to $\boldsymbol{n}$ and contains $\boldsymbol{p}$ is the smooth surface implicitly defined by the surface generating function f.

