6 Orientation – change of basis and rotation

6.1 Overview

Orientation, change of basis operations, and rotation operations are closely related concepts that are important in many different application domains. Unfortunately, the terminology and notation used to describe these concepts is diverse, and often inconsistent, causing confusion and errors.

A *change of basis* operation inputs a position-vector expressed in one orthonormal frame and outputs a position-vector for the same position expressed in a different orthonormal frame. Change of basis operations are the foundation of coordinate conversion and transformation computations (see Clause 10). Change of basis operations are normatively defined in 6.2.

The *orientation* of a rigid spatial object describes its angular displacement, or attitude, with respect to a reference, and is part of its state, along with its position and other spatial characteristics. The orientation of one orthonormal frame with respect to a second orthonormal frame is the directed angular relationship between them, with the second orthonormal frame serving as the reference. The specification of an orientation is important in many application domains including graphical rendering, interpretation of imagery, analysis of directional sensor data, robotics, vehicle aspect tracking, and the computations of direction and trajectory. Orientation is normatively defined and related to both change of basis and rotation operations in 6.3.

A *rotation* operation inputs a position-vector expressed in an orthonormal frame and outputs a position-vector for a different position that is rotated about a specified axis by a specified angle, expressed in the same orthonormal frame. Rotation operations are critical to the representation of motion, force, and dynamics in many application domains, including mechanics, aviation, and astronomy. Rotation can be interpreted in terms of the physical movement of objects, abstract geometry, or mathematical operations including change of basis operations. Rotation operations are normatively defined in 6.4.

Change of basis and rotation operators are summarized in 6.5. Rotations and orientations are commonly expressed in various forms, including axis-angle, matrices, Euler angles, and quaternions. These forms are normatively defined in 6.6. Conversions between these forms are normatively defined in 6.7.

6.2 Change of basis

6.2.1 Overview

Within a Euclidean vector space, change of basis operations allow a vector expressed in terms of a given basis to be re-expressed in terms of a different basis. Change of basis operations are used in many types of matrix computations. In this document, change of basis operations are used to express position-vectors, directions, and vector quantities in terms of different orthonormal frames.

6.2.2 Change of basis operations

A *change of basis* operation acts on a position-vector expressed in one orthonormal frame and produces the equivalent position-vector expressed in terms of a different orthonormal frame. In general, a change of basis operation can include an angular component and, when the frame origins differ, a positional displacement component. In some contexts, a change of basis operation can also include a scaling component (see 7.3.2).

$\boldsymbol{E}$ and $\boldsymbol{F}$ are two right-handed 3D orthonormal frames with respective basis vectors specified as $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. There is interest in computing the coordinate of a position-vector provided in one frame in terms of the other frame. When the origins of the two frames are different, denote the respective frame origins by $\boldsymbol{O_E}$ and $\boldsymbol{O_F}$. The vector from the origin of frame $\boldsymbol{E}$ to the origin of frame $\boldsymbol{F}$ is $\overrightarrow{\boldsymbol{O_E O_F}}$, which is the origin of frame $\boldsymbol{F}$ expressed in terms of frame $\boldsymbol{E}$. The inverse vector from the origin of frame $\boldsymbol{F}$ to the origin of frame $\boldsymbol{E}$ is $\overrightarrow{\boldsymbol{O_F O_E}}$, which is the origin of frame $\boldsymbol{E}$ expressed in terms of frame $\boldsymbol{F}$.

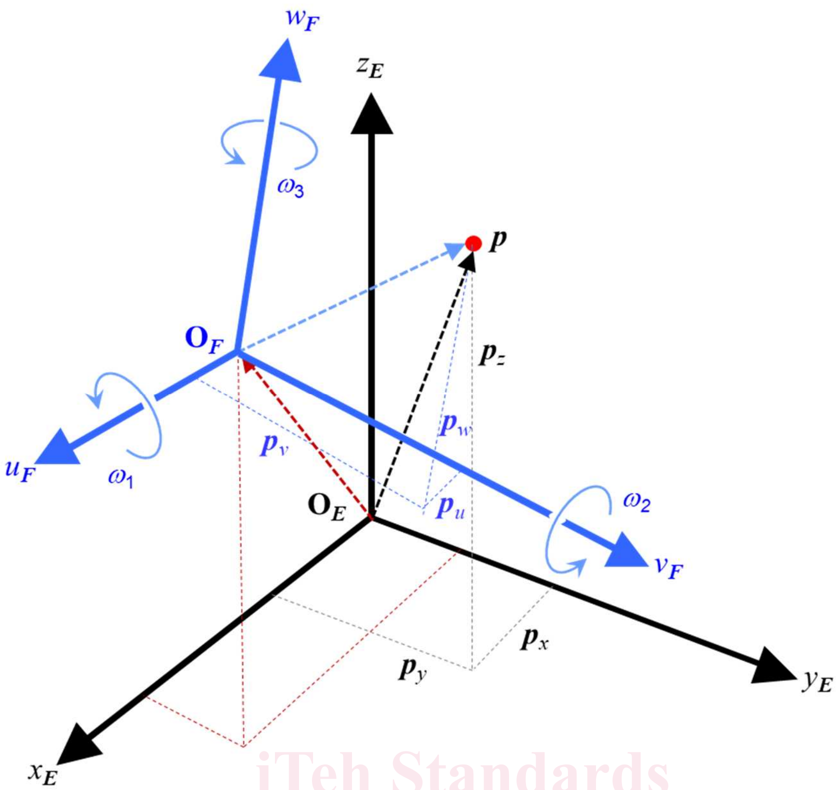

Figure 6.1 — Change of basis relationships

As Figure 6.1 illustrates, the position-vector $\boldsymbol{p}$ can be expressed with respect to the origin of frame $\boldsymbol{E}$ as $\overrightarrow{\boldsymbol{O_E p}}$, As Figure 6.1 further illustrates, $\boldsymbol{p}$ can also be expressed with respect to the origin of frame $\boldsymbol{E}$ as the vector sum $\overrightarrow{\boldsymbol{O_E p}} = \overrightarrow{\boldsymbol{O_E O_F}} + \overrightarrow{\boldsymbol{O_F p}}$. Thus, $\boldsymbol{p}$ can be expressed with respect to the origin of frame $\boldsymbol{F}$ as:

$$\overrightarrow{\boldsymbol{O_F p}}. = \overrightarrow{\boldsymbol{O_E p}} - \overrightarrow{\boldsymbol{O_E O_F}}$$

or, reversing the direction of $\overrightarrow{\boldsymbol{O_E O_F}}$:

$$\overrightarrow{\boldsymbol{O_F p}}. = \overrightarrow{\boldsymbol{O_F O_E}} + \overrightarrow{\boldsymbol{O_E p}}$$

The position-vector $\boldsymbol{p}$ represented in terms of frame $\boldsymbol{E}$ and denoted by $\boldsymbol{p_E}$, is the same vector as $\overrightarrow{\boldsymbol{O_E p}}$. Similarly, the position-vector $\boldsymbol{p}$ represented in terms of frame $\boldsymbol{F}$ and denoted by $\boldsymbol{p_F}$, is the same vector as $\overrightarrow{\boldsymbol{O_F p}}$. The transformation operation that re-expresses $\boldsymbol{p_E}$ in terms of frame $\boldsymbol{F}$ is:

$$\boldsymbol{p_F} = \overrightarrow{\boldsymbol{O_F O_E}} + \boldsymbol{\Omega_{F \leftarrow E} p_E},$$

where $\overrightarrow{\boldsymbol{O_F O_E}}$ denotes the positional displacement component and $\boldsymbol{\Omega_{F \leftarrow E}}$ denotes the angular displacement component. The direction of the positional displacement vector is from the origin of the target frame to the origin of the source frame. The inverse transformation operation that re-expresses $\boldsymbol{p_F}$ in terms of frame $\boldsymbol{E}$ is:

$$\boldsymbol{p_E} = \overrightarrow{\boldsymbol{O_E O_F}} + \boldsymbol{\Omega_{E \leftarrow F} p_F}.$$

If frames $\boldsymbol{E}$ and $\boldsymbol{F}$ have a common origin, there is no positional displacement component, and thus $\boldsymbol{p_E}$ can be re-expressed in terms of frame $\boldsymbol{F}$ using only the angular displacement component:

$$\boldsymbol{p_F} = \boldsymbol{\Omega_{F \leftarrow E}}\, \boldsymbol{p_E}.$$

The inverse transformation is:

$$\boldsymbol{p_E} = \boldsymbol{\Omega_{E \leftarrow F}}\, \boldsymbol{p_F}.$$

Throughout the remainder of this clause, unless otherwise specified, a common origin for both frames is assumed. Thus, the phrase *change of basis* is used to refer to only the angular displacement component of the operation, denoted by $\boldsymbol{\Omega}$ with appropriate subscripts.

For a position-vector $\boldsymbol{p}$, the frame $\boldsymbol{E}$ coordinate for $\boldsymbol{p}$ with respect to the common origin is $\left(p_x, p_y, p_z\right)_{\boldsymbol{E}}$, where each scalar value is the dot product of the position-vector with one of the basis vectors of the orthonormal frame:

$$p_x = \boldsymbol{p} \bullet \boldsymbol{x},\ p_y = \boldsymbol{p} \bullet \boldsymbol{y},\ p_z = \boldsymbol{p} \bullet \boldsymbol{z}.$$

Similarly, the frame $\boldsymbol{F}$ coordinate for $\boldsymbol{p}$ is $(p_{\boldsymbol{u}}, p_{\boldsymbol{v}}, p_{\boldsymbol{w}})_{\boldsymbol{F}}$, where

$$p_{\boldsymbol{u}} = \boldsymbol{p} \bullet \boldsymbol{u},\ p_{\boldsymbol{v}} = \boldsymbol{p} \bullet \boldsymbol{v},\ p_{\boldsymbol{w}} = \boldsymbol{p} \bullet \boldsymbol{w}.$$

The linear combination with respect to frame $\boldsymbol{E}$ can be written as:

$$\boldsymbol{p} = p_x \boldsymbol{x} + p_y \boldsymbol{y} + p_z \boldsymbol{z}.$$

Using this expression for $\boldsymbol{p}$, the $\boldsymbol{F}$ frame coordinate components of $\boldsymbol{p}$ become:

$$\begin{aligned}
p_{\boldsymbol{u}} &= \boldsymbol{p} \bullet \boldsymbol{u} = \left(p_x \boldsymbol{x} + p_y \boldsymbol{y} + p_z \boldsymbol{z}\right) \bullet \boldsymbol{u} = p_x \boldsymbol{x} \bullet \boldsymbol{u} + p_y \boldsymbol{y} \bullet \boldsymbol{u} + p_z \boldsymbol{z} \bullet \boldsymbol{u} \\
p_{\boldsymbol{v}} &= \boldsymbol{p} \bullet \boldsymbol{v} = \left(p_x \boldsymbol{x} + p_y \boldsymbol{y} + p_z \boldsymbol{z}\right) \bullet \boldsymbol{v} = p_x \boldsymbol{x} \bullet \boldsymbol{v} + p_y \boldsymbol{y} \bullet \boldsymbol{v} + p_z \boldsymbol{z} \bullet \boldsymbol{v} \\
p_{\boldsymbol{w}} &= \boldsymbol{p} \bullet \boldsymbol{w} = \left(p_x \boldsymbol{x} + p_y \boldsymbol{y} + p_z \boldsymbol{z}\right) \bullet \boldsymbol{w} = p_x \boldsymbol{x} \bullet \boldsymbol{w} + p_y \boldsymbol{y} \bullet \boldsymbol{w} + p_z \boldsymbol{z} \bullet \boldsymbol{w}
\end{aligned}$$

The matrix form of this system of linear equations is:

$$\begin{bmatrix} p_{\boldsymbol{u}} \\ p_{\boldsymbol{v}} \\ p_{\boldsymbol{w}} \end{bmatrix}_{\boldsymbol{F}} = \begin{bmatrix} \boldsymbol{x} \bullet \boldsymbol{u} & \boldsymbol{y} \bullet \boldsymbol{u} & \boldsymbol{z} \bullet \boldsymbol{u} \\ \boldsymbol{x} \bullet \boldsymbol{v} & \boldsymbol{y} \bullet \boldsymbol{v} & \boldsymbol{z} \bullet \boldsymbol{v} \\ \boldsymbol{x} \bullet \boldsymbol{w} & \boldsymbol{y} \bullet \boldsymbol{w} & \boldsymbol{z} \bullet \boldsymbol{w} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}_{\boldsymbol{E}}, \text{ or}$$

$$\begin{bmatrix} p_{\boldsymbol{u}} \\ p_{\boldsymbol{v}} \\ p_{\boldsymbol{w}} \end{bmatrix}_{\boldsymbol{F}} = \boldsymbol{M_{F \leftarrow E}} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}_{\boldsymbol{E}}, \text{ or}$$

$$\boldsymbol{p_F} = \boldsymbol{M_{F \leftarrow E}} \boldsymbol{p_E}$$

This matrix multiplication operation $\boldsymbol{M_{F \leftarrow E}}$ is equivalent to the change of basis operation $\boldsymbol{\Omega_{F \leftarrow E}}$:

$$\boldsymbol{p_F} = \boldsymbol{\Omega_{F \leftarrow E}}\, \boldsymbol{p_E}.$$

Since both frames are orthonormal and have a common origin, the relationship also represents the projection of the basis vectors of one frame onto the basis vectors of the other frame. The columns of $\boldsymbol{\Omega_{F \leftarrow E}}$ are the $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ basis vectors in terms of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ coordinate-components while the rows (or columns of the transpose matrix $\boldsymbol{\Omega_{E \leftarrow F}}$) are the $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ basis vectors in terms of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ coordinate-components.

$$\begin{aligned}
\boldsymbol{\Omega_{F \leftarrow E}} &= \boldsymbol{\Omega_{E \leftarrow F}^{-1}} = \begin{bmatrix} \boldsymbol{x} \bullet \boldsymbol{u} & \boldsymbol{y} \bullet \boldsymbol{u} & \boldsymbol{z} \bullet \boldsymbol{u} \\ \boldsymbol{x} \bullet \boldsymbol{v} & \boldsymbol{y} \bullet \boldsymbol{v} & \boldsymbol{z} \bullet \boldsymbol{v} \\ \boldsymbol{x} \bullet \boldsymbol{w} & \boldsymbol{y} \bullet \boldsymbol{w} & \boldsymbol{z} \bullet \boldsymbol{w} \end{bmatrix} = \boldsymbol{M_{F \leftarrow E}} \\
\boldsymbol{\Omega_{E \leftarrow F}} &= \boldsymbol{\Omega_{F \leftarrow E}^{-1}} = \begin{bmatrix} \boldsymbol{u} \bullet \boldsymbol{x} & \boldsymbol{v} \bullet \boldsymbol{x} & \boldsymbol{w} \bullet \boldsymbol{x} \\ \boldsymbol{u} \bullet \boldsymbol{y} & \boldsymbol{v} \bullet \boldsymbol{y} & \boldsymbol{w} \bullet \boldsymbol{y} \\ \boldsymbol{u} \bullet \boldsymbol{z} & \boldsymbol{v} \bullet \boldsymbol{z} & \boldsymbol{w} \bullet \boldsymbol{z} \end{bmatrix} = \boldsymbol{M_{E \leftarrow F}}
\end{aligned} \tag{6.1}$$

These operators define the change of basis relationship between the two frames $\boldsymbol{E}$ and $\boldsymbol{F}$, allowing position-vector representations to be converted from one to the other, in either direction.

6.2.3 Direction cosine matrix

Expressing the basis vectors of one frame in terms of the other frame provides the relationship between the two frames. One way to express the relationship is based on the cosine of the angle between each basis vector of a frame and all basis vectors of the other frame. Since basis vectors are unit vectors, each dot product in Equation 6.1 is the cosine of the angle (θ) between the two indicated vectors (see A.2). A total of nine cosine values are required to describe the full relationship between two 3D frames. Arranged as a matrix, for frames $\boldsymbol{E}$ and $\boldsymbol{F}$ the nine cosine values are represented as:

$$\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}} = \boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}^{-1} = \begin{bmatrix} \cos(\boldsymbol{\theta}_{xu}) & \cos(\boldsymbol{\theta}_{yu}) & \cos(\boldsymbol{\theta}_{zu}) \\ \cos(\boldsymbol{\theta}_{xv}) & \cos(\boldsymbol{\theta}_{yv}) & \cos(\boldsymbol{\theta}_{zv}) \\ \cos(\boldsymbol{\theta}_{xw}) & \cos(\boldsymbol{\theta}_{yw}) & \cos(\boldsymbol{\theta}_{zw}) \end{bmatrix}$$

$$\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}} = \boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}^{-1} = \begin{bmatrix} \cos(\boldsymbol{\theta}_{ux}) & \cos(\boldsymbol{\theta}_{vx}) & \cos(\boldsymbol{\theta}_{wx}) \\ \cos(\boldsymbol{\theta}_{uy}) & \cos(\boldsymbol{\theta}_{vy}) & \cos(\boldsymbol{\theta}_{wy}) \\ \cos(\boldsymbol{\theta}_{uz}) & \cos(\boldsymbol{\theta}_{vz}) & \cos(\boldsymbol{\theta}_{wz}) \end{bmatrix}$$

The first matrix expresses the basis vectors of frame $\boldsymbol{E}$ in terms of frame $\boldsymbol{F}$. The second matrix is the inverse and expresses the unit vectors of $\boldsymbol{F}$ in terms of $\boldsymbol{E}$. Each of these two matrices are often referred to as a *direction cosine matrix*. It is noted that the sum of the square of the values in each column is one.

6.2.4 Consecutive change of basis

Given three right-handed orthonormal frames $\boldsymbol{D}$, $\boldsymbol{E}$, and $\boldsymbol{F}$ with a common origin and the change of basis operators $\boldsymbol{\Omega}_{\boldsymbol{D}\leftarrow\boldsymbol{E}}$ and $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and their inverses $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{D}}$ and $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$, the change of basis operators $\boldsymbol{\Omega}_{\boldsymbol{D}\leftarrow\boldsymbol{F}}$ and $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{D}}$ are the compositions:

$$\boldsymbol{\Omega}_{\boldsymbol{D}\leftarrow\boldsymbol{F}} = \boldsymbol{\Omega}_{\boldsymbol{D}\leftarrow\boldsymbol{E}} \circ \boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$$

$$\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{D}} = \boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}} \circ \boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{D}}$$

This result generalizes to a chain of orthonormal frames with different origins. For a chain of length N, denote the n^{th} frame by $\boldsymbol{F}_{\boldsymbol{n}}$, its origin by $\boldsymbol{O}_{\boldsymbol{n}}$ and the displacement vector from the $\boldsymbol{F}_{\boldsymbol{j}}$ origin to the $\boldsymbol{F}_{\boldsymbol{n}}$ origin by $\overrightarrow{\boldsymbol{O}_{\boldsymbol{j}}\boldsymbol{O}_{\boldsymbol{n}}}$ for $1 \le j, n \le N$. For any $1 \le j < n < N$, the change of basis operator, along with positional components, from frame $\boldsymbol{F}_{\boldsymbol{j}}$ to frame $\boldsymbol{F}_{\boldsymbol{n}}$ is denoted by $\overrightarrow{\boldsymbol{O}_{\boldsymbol{n}}\boldsymbol{O}_{\boldsymbol{j}}} + \boldsymbol{\Omega}_{\boldsymbol{n}\leftarrow\boldsymbol{j}}$.

For a chain of frames from $\boldsymbol{F}_{\boldsymbol{j}}$ to $\boldsymbol{F}_{\boldsymbol{n}}$, the composition of the consecutive chain of operations is:

$$\overrightarrow{\boldsymbol{O}_{\boldsymbol{n}}\boldsymbol{O}_{\boldsymbol{j}}} + \boldsymbol{\Omega}_{\boldsymbol{n}\leftarrow\boldsymbol{j}} = \left(\overrightarrow{\boldsymbol{O}_{\boldsymbol{n}}\boldsymbol{O}_{\boldsymbol{m}}} + \cdots + \overrightarrow{\boldsymbol{O}_{\boldsymbol{k}}\boldsymbol{O}_{\boldsymbol{j}}}\right) + \left(\boldsymbol{\Omega}_{\boldsymbol{n}\leftarrow\boldsymbol{m}} \circ \ldots \circ \boldsymbol{\Omega}_{\boldsymbol{k}\leftarrow\boldsymbol{j}}\right) = \overrightarrow{\boldsymbol{O}_{\boldsymbol{n}}\boldsymbol{O}_{\boldsymbol{j}}} + \left(\boldsymbol{\Omega}_{\boldsymbol{n}\leftarrow\boldsymbol{m}} \circ \ldots \circ \boldsymbol{\Omega}_{\boldsymbol{k}\leftarrow\boldsymbol{j}}\right).,$$

since the vector sum of the chain of vectors $\left(\overrightarrow{\boldsymbol{O}_{\boldsymbol{n}}\boldsymbol{O}_{\boldsymbol{m}}} + \cdots + \overrightarrow{\boldsymbol{O}_{\boldsymbol{k}}\boldsymbol{O}_{\boldsymbol{j}}}\right)$ is equivalent to the single vector $\overrightarrow{\boldsymbol{O}_{\boldsymbol{n}}\boldsymbol{O}_{\boldsymbol{j}}}$.

6.2.5 Equivalence of change of basis and rotation operators

The common origin of the right-handed orthonormal frames $\boldsymbol{E}$ and $\boldsymbol{F}$ is a fixed point of the operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$. Euler's rotation theorem states that any length-preserving transformation of 3D space that has at least one point fixed under the transformation is equivalent to a single rotation about an axis that passes through the fixed point. This implies that $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ is equivalent to a rotation operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ (see 6.4.2.1), where $\boldsymbol{n}$ is the axis of rotation passing through the origin and θ is the rotation angle. This operator rotates a position-vector $\boldsymbol{p}$ to $\boldsymbol{p}' = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{p})$. The equivalence of $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ is shown in A.12.

Applying $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ to the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of frame $\boldsymbol{E}$ yields the basis vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ of frame $\boldsymbol{F}$:

$$\boldsymbol{u} = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{x})$$

$$\boldsymbol{v} = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{y})$$

$$\boldsymbol{w} = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{z})$$

The rotation operation $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ can also be designated as $\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$. Hence, the orientation of object-frame $\boldsymbol{F}$ with respect to reference-frame $\boldsymbol{E}$ is realised by both the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and the rotation operator $\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$. Thus, the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and the rotation operation $\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$ are equivalent to each other:

$$\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}} = \boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$$

The difference between operators $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and $\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$ is in the interpretation of the output of the operation as either the change of basis for any position-vector in terms of the bases of $\boldsymbol{F}$ and $\boldsymbol{E}$ or as the rotation of that position vector about axis $\boldsymbol{n}$ though angle θ. Applying this rotation to each of the basis vectors of frame $\boldsymbol{E}$ yields the basis vectors of frame $\boldsymbol{F}$, in effect rotating frame $\boldsymbol{F}$ away from alignment with frame $\boldsymbol{E}$.

The direction cosine matrix that corresponds to the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and the rotation matrix that corresponds to the rotation operator $\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$ are therefore also equivalent:

$$\begin{bmatrix} \boldsymbol{u}\bullet\boldsymbol{x} & \boldsymbol{v}\bullet\boldsymbol{x} & \boldsymbol{w}\bullet\boldsymbol{x} \\ \boldsymbol{u}\bullet\boldsymbol{y} & \boldsymbol{v}\bullet\boldsymbol{y} & \boldsymbol{w}\bullet\boldsymbol{y} \\ \boldsymbol{u}\bullet\boldsymbol{z} & \boldsymbol{v}\bullet\boldsymbol{z} & \boldsymbol{w}\bullet\boldsymbol{z} \end{bmatrix} = \begin{bmatrix} \cos(\boldsymbol{\theta}_{ux}) & \cos(\boldsymbol{\theta}_{vx}) & \cos(\boldsymbol{\theta}_{wx}) \\ \cos(\boldsymbol{\theta}_{uy}) & \cos(\boldsymbol{\theta}_{vy}) & \cos(\boldsymbol{\theta}_{wy}) \\ \cos(\boldsymbol{\theta}_{uz}) & \cos(\boldsymbol{\theta}_{vz}) & \cos(\boldsymbol{\theta}_{wz}) \end{bmatrix}$$

6.2.6 Change of basis and orientation

The change of basis operators $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$ express the bidirectional angular relationship between the two orthonormal frames $\boldsymbol{E}$ and $\boldsymbol{F}$. This bidirectional relationship is expressed in terms of the angles between each pair of basis vectors in the direction cosine matrices. Thus, the orientation of orthonormal frame $\boldsymbol{F}$ with respect to orthonormal frame $\boldsymbol{E}$ is represented by the direction cosine matrix that corresponds to the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$. Similarly, the orientation of orthonormal frame $\boldsymbol{E}$ with respect to orthonormal frame $\boldsymbol{F}$ is represented by the direction cosine matrix that corresponds to the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$.

6.3 Orientation

6.3.1 Overview

The orientation of a rigid object describes its angular displacement, or attitude, with respect to a reference. When the object is represented by an orthonormal frame attached to the object, the orientation of the object is represented by the angular displacement of the object's frame with respect to an orthonormal reference frame. This angular displacement can be specified in terms of either: 1) a change of basis that converts a coordinate from the object's frame to the reference frame, or 2) a rotation of the object's frame away from alignment with the reference frame.

Specification of and computations with orientations are defined with respect to orthonormal frames (see 5.2.2). An orthonormal frame serving in the role of an orientation reference is termed a *reference-frame*. An orthonormal frame that is, conceptually or physically, rigidly attached to an object of interest is termed an *object-frame*.

Object-frame attachment choices have significant effects on computational results and will affect interoperability if not clearly specified. An object-frame can be attached to an object in many ways. The choice of object-frame origin attachment point and alignment of axis directions is highly dependent on the application domain and is not addressed in this document.

There are infinitely many ways to attach the origin of an orthonormal frame to an object. The origin can be located at any point within the spatial object of interest, at any point on its surface, or at any point nearby in space. Common selections include the centre of mass of the object, its geometric centre, a corner of the object (assuming it has corners), or its bounding volume such that the object is completely within the first octant.

Given a selected origin, there are infinitely many ways to orient the basis vectors of the orthonormal frame. If the object is a celestial body, the basis vectors might be aligned with its rotational axis, its magnetic field axis, or the direction of the closest star (such as the Sun). If the object is a vehicle, such as an aircraft, the basis vectors might be aligned based on its direction of forward motion or other common reference orientations. If the